Management Science
Linear Programming

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Session Objectives

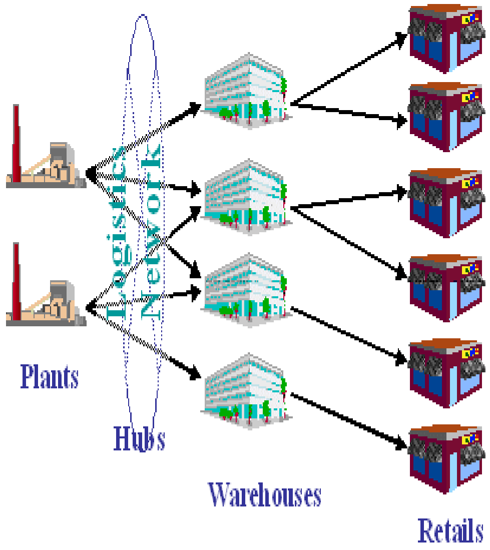
- Understand the basic assumptions and properties of linear programming (LP)
- Graphically solve a LP problem that has only two variables
- Understand special issues in LP such as infeasibility, unboundedness, redundancy, and alternative optimal solutions
- Understand the simplex method
- Understand matrix methods
- Understand the role of sensitivity analysis
- Use Excel spreadsheets to solve LP problems

Introduction

- Many management decisions involve trying to make the most effective use of limited resources
  - Machinery, labor, money, time, warehouse space, raw materials
- This often involves a large class of problems where the outcome is a linear combination of the inputs ($z = ax_1+bx_2+\ldots$), and the inputs are limited ($x_1 < \ldots$)
- The objective is to maximize (or minimize) the outcome (z) by setting each variable (x_i) to an optimal value

Distribution Problem

- Each plant has a certain production capacity and per unit production cost
- Each warehouse has a certain customer demand
- Transportation cost varies from origin plant to warehouse
- **How much should be supplied from each plant to each warehouse? What needs to be considered ?**



Distribution Problem (con't)

- Want to make the most use of the plants with the lowest production cost, but the transportation costs may be too high
- Similar to problem many US companies face with decisions on outsourcing, since foreign production costs are lower, but it costs more to transport goods back to US

Production Problem

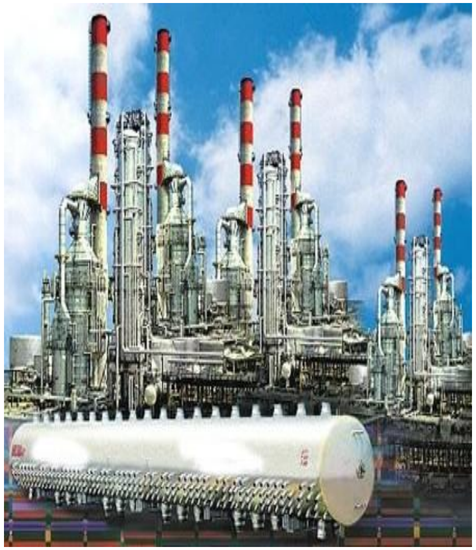
- Consider the problem of a number of plants each with a certain range in production rate and associated costs which are higher when the plant goes into overtime
- Production has to meet sales forecasts in a number of periods
- A production schedule must be found to minimize the total cost over the time periods
- **What needs to be considered ?**

Production Problem (con't)

- If material is produced ahead of time, there are inventory costs
- There are overtime costs involved with high production rates in plants

Blending Problem

- Consider the problem where there are a number of raw materials available at various costs and in various quantities
- The problem is to find the “recipe” to be used which will yield a satisfactory product (must have at least so much but not more than so much of each raw material) at the minimum cost or maximum profit
- This is a common problem in the production of animal feeds, and also in the blending of crude oil fractions to satisfy product demand



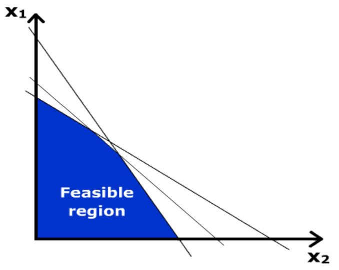
Requirements of a Linear Programming Problem

- LP has been applied in many areas over the past 50+ years
- All LP problems have 4 properties in common
  1. All problems seek to *maximize* or *minimize* some quantity (the *objective function*)
  2. The presence of restrictions or *constraints* that limit the degree to which we can pursue our objective
  3. There must be alternative courses of action to choose from
  4. The objective and constraints in problems must be expressed in terms of *linear* equations or inequalities

LP Properties and Assumptions

PROPERTIES OF LINEAR PROGRAMS

1. One objective function
2. One or more constraints
3. Alternative courses of action (values of the variables)
4. Objective function and constraints are linear
5. Certainty of coefficients (deterministic model)
6. Divisibility (variables need not be whole numbers)
7. Nonnegative variables



Formulating LP Problems

- Formulating a linear program involves developing a mathematical model to represent the managerial problem
- The steps in formulating a linear program are
  1. Completely understand the managerial problem being faced
  2. Identify the objective and constraints
  3. Define the decision variables
  4. Use the decision variables to write mathematical expressions for the objective function and the constraints

Variables

- In each of these problems there are several variables:
  - Quantities to be shipped from each plant
  - Amount to be produced at each plant in each time period
  - Quantity of each raw material to be used in the recipe
- These can be written as X_{ij}
  - Such as the amount to be shipped from the i'th plant to the j'th warehouse

Objective or Cost Function

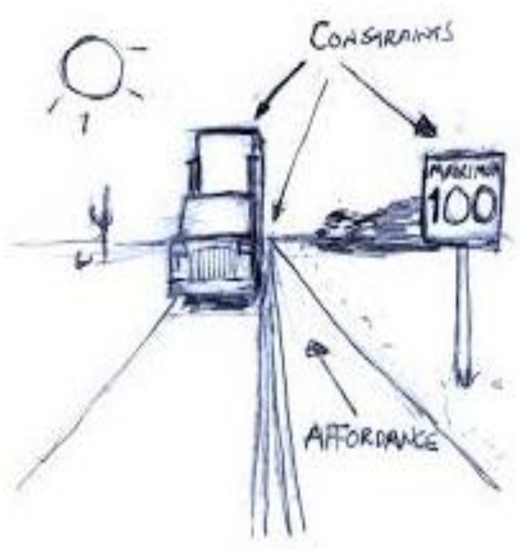
- The “cost” of the result depends upon a linear combination of the variables
- This means that there are known “cost coefficients” c_{ij}
  - Such as the cost per unit quantity to ship from plant i to warehouse j
- The total cost is:
  - $\sum\sum c_{ij}x_{ij}$
  - Where the first sum is over the plants, and the second sum is over the warehouses

Non-Linear Cases

- The problem would not be linear if:
  - The cost depended upon the product of two or more variables
  - Some power of the variable was involved as the square, square root, reciprocal
  - Other functions were involved as logs, sine's, etc.
  - Other combinations or extensions of these things
- In these cases, a non-linear solution method must be used

Constraints

- Each of the variables can only take on values between certain limits
- For example, the plants cannot ship more than their production limit, or a warehouse cannot receive more than it ships, or the amount of a certain raw material must be at least so much of the final recipe



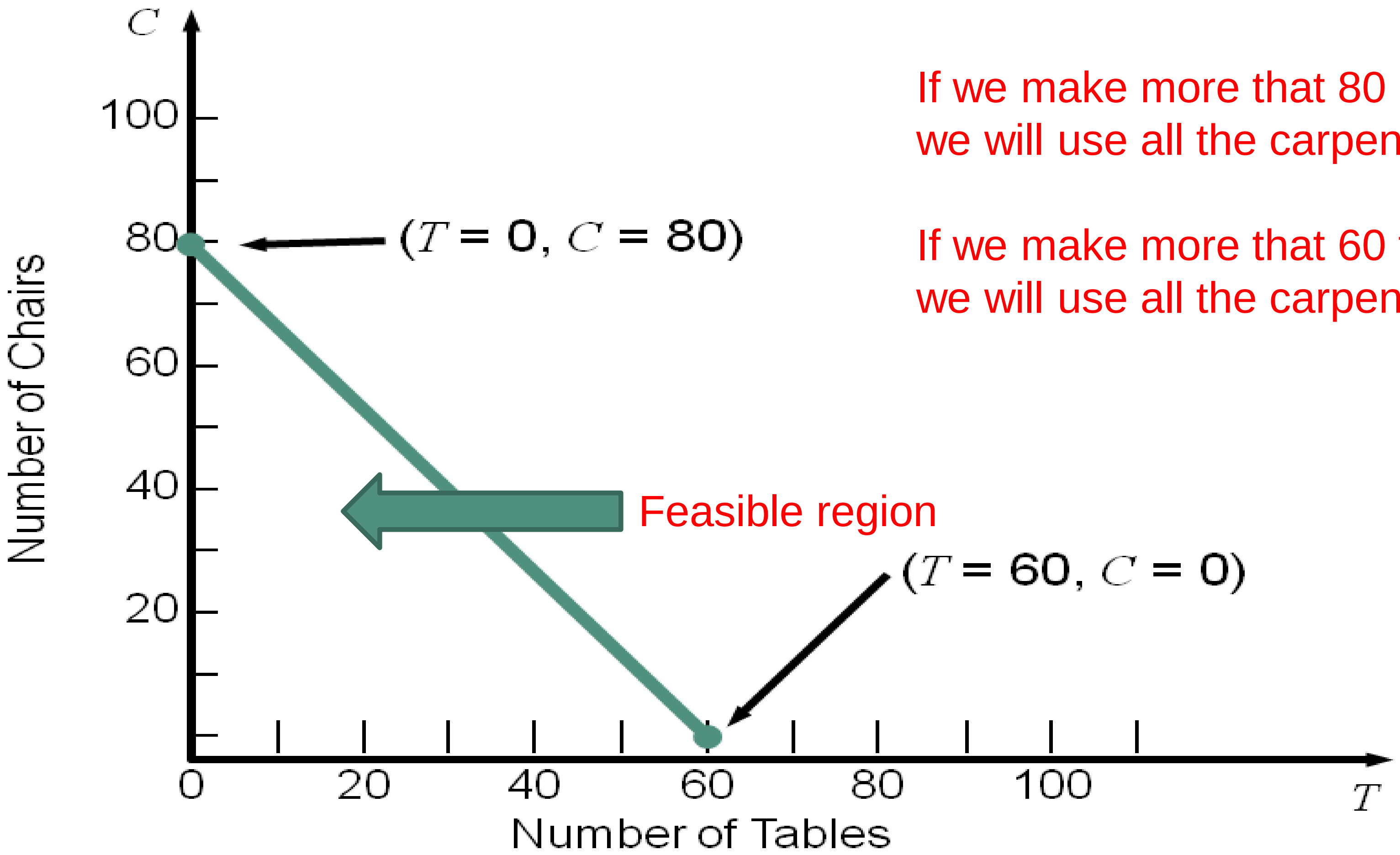
Example Problem – Product Mix

- One of the most common LP applications is the *product mix problem*
- Two or more products are produced using limited resources such as personnel, machines, and raw materials
- The profit that the firm seeks to maximize is based on the profit contribution per unit of each product
- The company would like to determine how many units of each product it should produce so as to <u>maximize overall profit</u> given its limited resources

Furniture Company

- A furniture company produces tables and chairs
- Each table takes 4 hours of carpentry and 2 hours of painting and varnishing
- Each chair requires 3 of carpentry and 1 hour of painting and varnishing
- There are 240 hours of carpentry time available and no limit on the amount of available time for painting and varnishing
- Each table yields a profit of $70 and each chair a profit of $50
- How many chairs and tables should the company produce ?

Single Constraint Model



If we make more that 80 chairs, we will use all the carpenter time

If we make more that 60 tables we will use all the carpenter time

Furniture Company (con't)

- Each table takes 4 hours of carpentry and 2 hours of painting and varnishing
- Each chair requires 3 of carpentry and 1 hour of painting and varnishing
- Each table yields a profit of $70 and each chair a profit of $50 → tables are more profitable
- Since there are only 240 hours of carpentry
  - If we made all chairs, then we could make 240/3 or 80 chairs, and sell the chairs for $50 * 80 = $4000
  - If we made all tables, then we could make 240/4 or 60 tables, and sell the tables for $70 * 60 = $4200
- Since there is only one constraint, the maximum profit is realized by making all tables

Furniture Company (con't)

- If we made 30 tables, then we would use 30 * 4 or 120 carpenter hours, and sell the tables for $70 * 30 = $2100
- If we then used the rest of the carpenter time (240-120=120) hours to made chairs, then we could make 120/3 or 40 chairs, and sell the chairs for $50 * 40 = $2000
- For a total profit of $4100
- There is no combination of tables and chairs that will do better than making all tables
- Since 70/4 (17.5) for tables or the ratio of table profit contribution divided by table resource usage is greater than 50/3 (or 16.67) for chair profit contribution divided by chair resource usage

Furniture Company with Second Constraint

- A furniture company produces tables and chairs
- Each table takes 4 hours of carpentry and 2 hours of painting and varnishing
- Each chair requires 3 of carpentry and 1 hour of painting and varnishing
- There are 240 hours of carpentry time available and 100 hours of painting and varnishing each time period
- Each table yields a profit of $70 and each chair a profit of $50

Furniture Company Data

The company wants to determine the best combination of tables and chairs to produce to reach the maximum profit

DEPARTMENT	HOURS REQUIRED TO PRODUCE 1 UNIT (T) TABLES	(C) CHAIRS	AVAILABLE HOURS THIS WEEK
Carpentry	**4**	**3**	**240**
Painting and varnishing	**2**	**1**	**100**
Profit per unit	**$70**	**$50**	

Furniture Company

- The objective is to:

  Maximize profit

- The constraints are:
  1. The hours of carpentry time used cannot exceed 240 hours per week
  2. The hours of painting and varnishing time used cannot exceed 100 hours per week
- The decision variables representing the actual decisions we will make are:

  T = number of tables to be produced per week

  C = number of chairs to be produced per week

Furniture Company

- LP objective function in terms of T and C:

  Maximize profit = $\$70T + \$50C$

- Mathematical relationships for the first constraint:
  - For carpentry, total time used is:

    (4 hours per table)(# tables produced)
    + (3 hours per chair)(# chairs produced)

Carpentry time used ≤ Carpentry time available

$4T + 3C \leq 240$ (hours of carpentry time)

Furniture Company

- Similarly,

  Painting and varnishing time used
  $\leq$ Painting and varnishing time available

  $2T + 1C \leq 100$ (hours of painting and varnishing time)

  This means that each table produced requires two hours of painting and varnishing time

- Both of these constraints restrict production capacity and affect total profit

Furniture Company

The values for T and C must be nonnegative

$T \geq 0$ (number of tables produced is greater than or equal to 0)

$C \geq 0$ (number of chairs produced is greater than or equal to 0)

The complete problem stated mathematically:

Maximize profit = \$70T + \$50C

subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

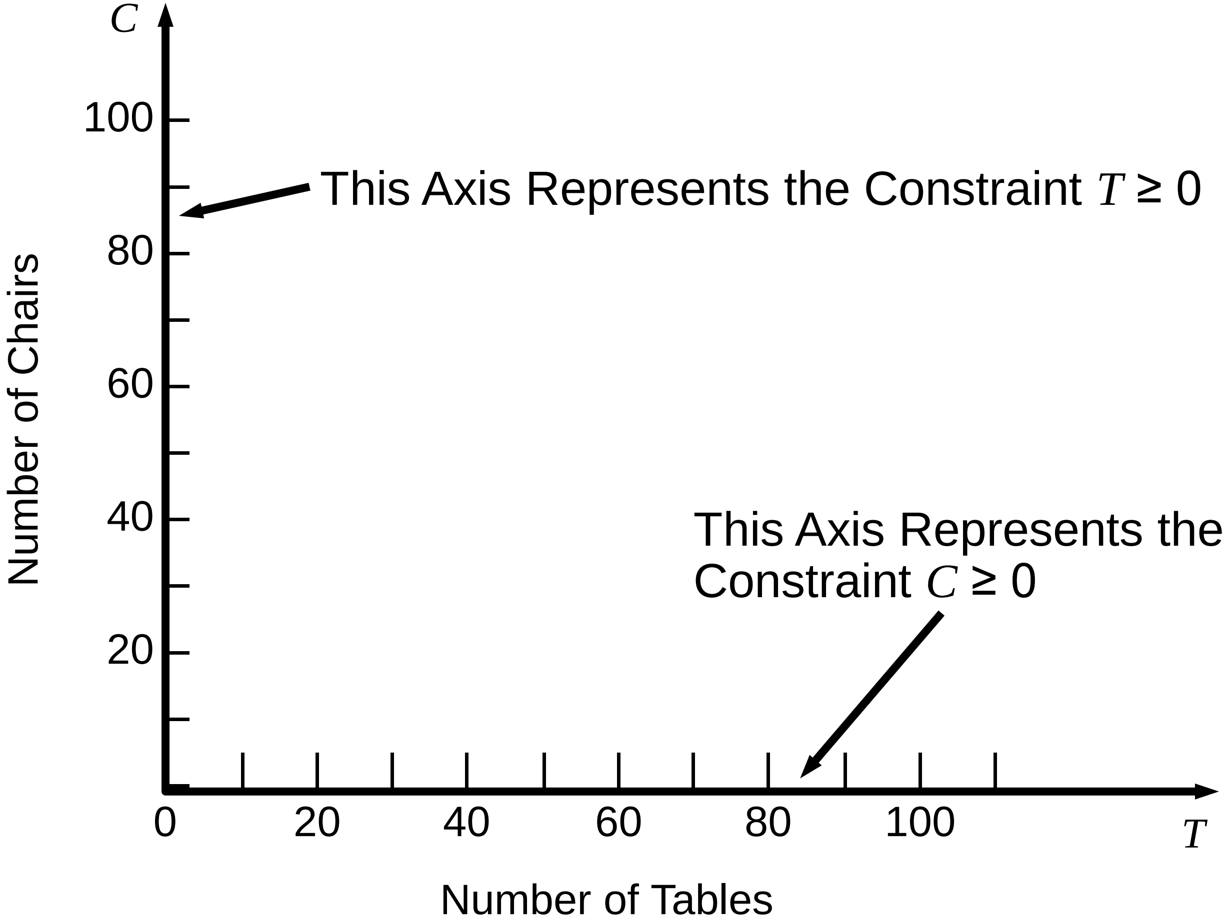
$T, C \geq 0$ (nonnegativity constraint)

Graphical Solution to an LP Problem

- The easiest way to solve a small LP problems is graphically
- The graphical method only works when there are just two decision variables
- When there are more than two variables, a more complex approach is needed as it is not possible to plot the solution on a two-dimensional graph
- The graphical method provides valuable insight into how other approaches work

Graphical Representation of a Constraint

Quadrant Containing All Positive Values



Graphical Representation of a Constraint (con't)

- The first step in solving the problem is to identify a set or region of feasible solutions
- To do this we plot each constraint equation on a graph
- We start by graphing the equality portion of the constraint equations:

$$4T + 3C = 240$$

- We solve for the axis intercepts and draw the line

Graphical Representation of a Constraint (con’t)

- For no tables, the carpentry constraint is:

$$4(0) + 3C = 240$$
$$3C = 240$$
$$C = 80$$

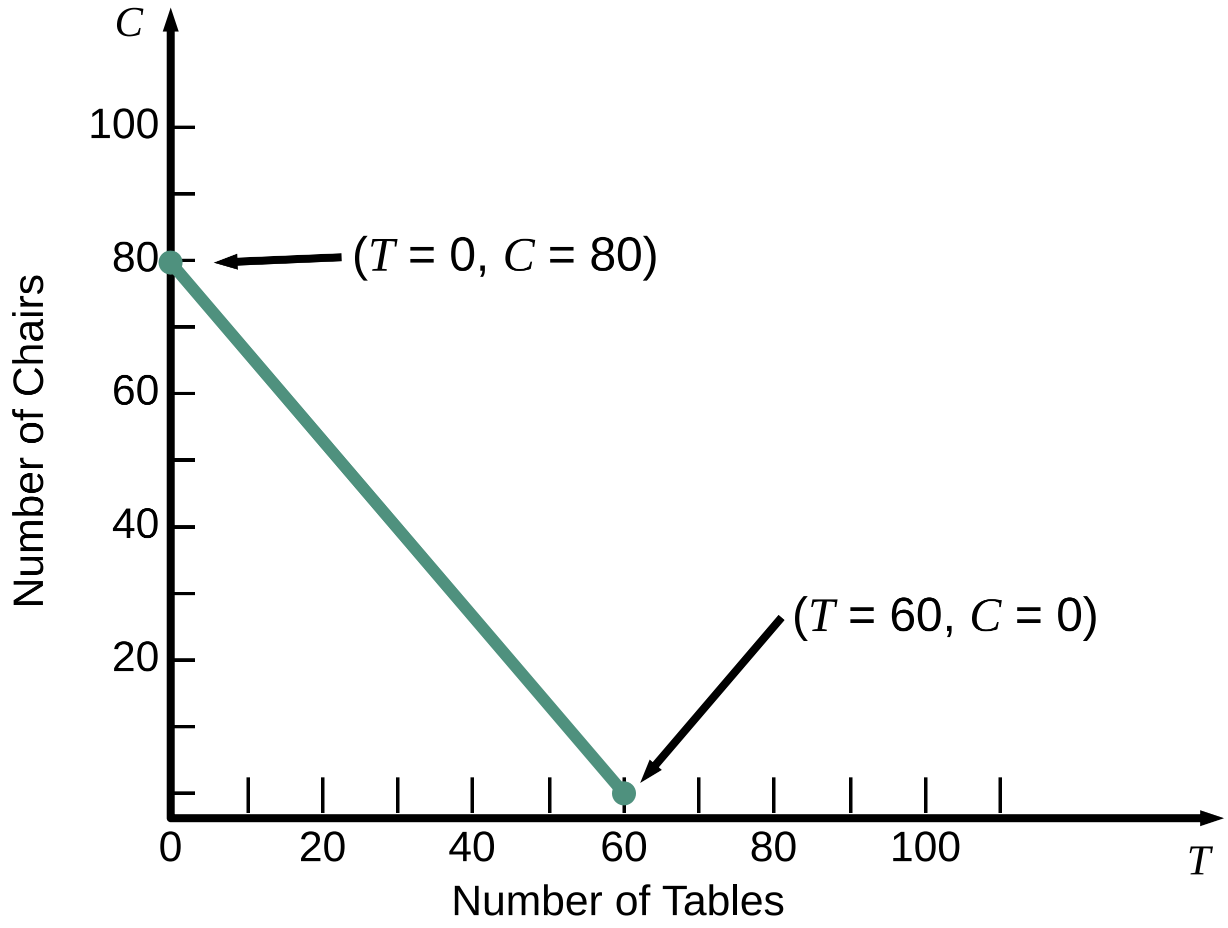
- Similarly for no chairs:

$$4T + 3(0) = 240$$
$$4T = 240$$
$$T = 60$$

- This line is shown on the following graph:

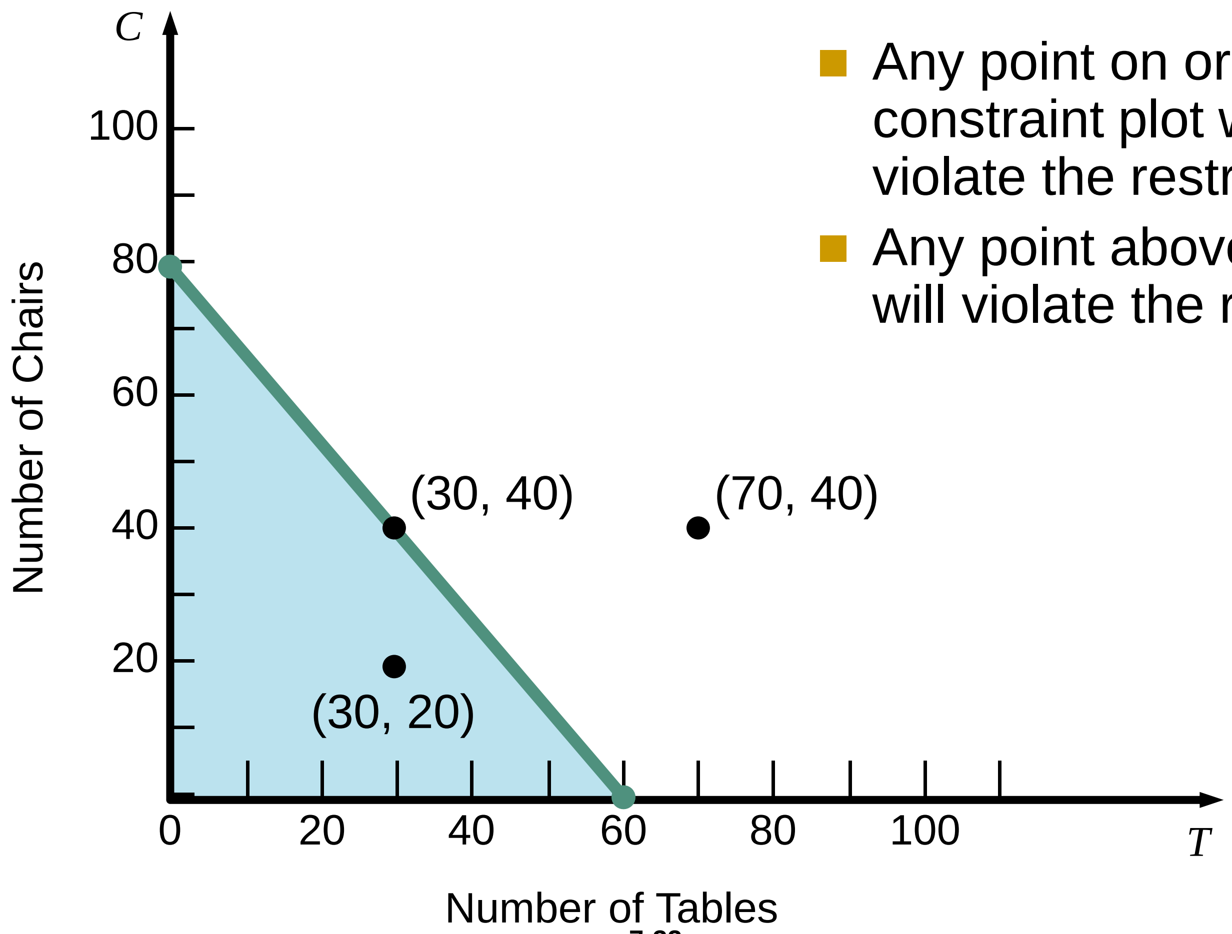
Graphical Representation of a Constraint (con't)

Graph of carpentry constraint equation



Graphical Representation of a Constraint (con't)

Region that Satisfies the Carpentry Constraint



- Any point on or below the constraint plot will not violate the restriction
- Any point above the plot will violate the restriction

Graphical Representation of a Constraint (con't)

- The point (30, 40) lies on the line and exactly satisfies the constraint

$$4(30) + 3(40) = 240$$

- The point (30, 20) lies below the line and satisfies the constraint

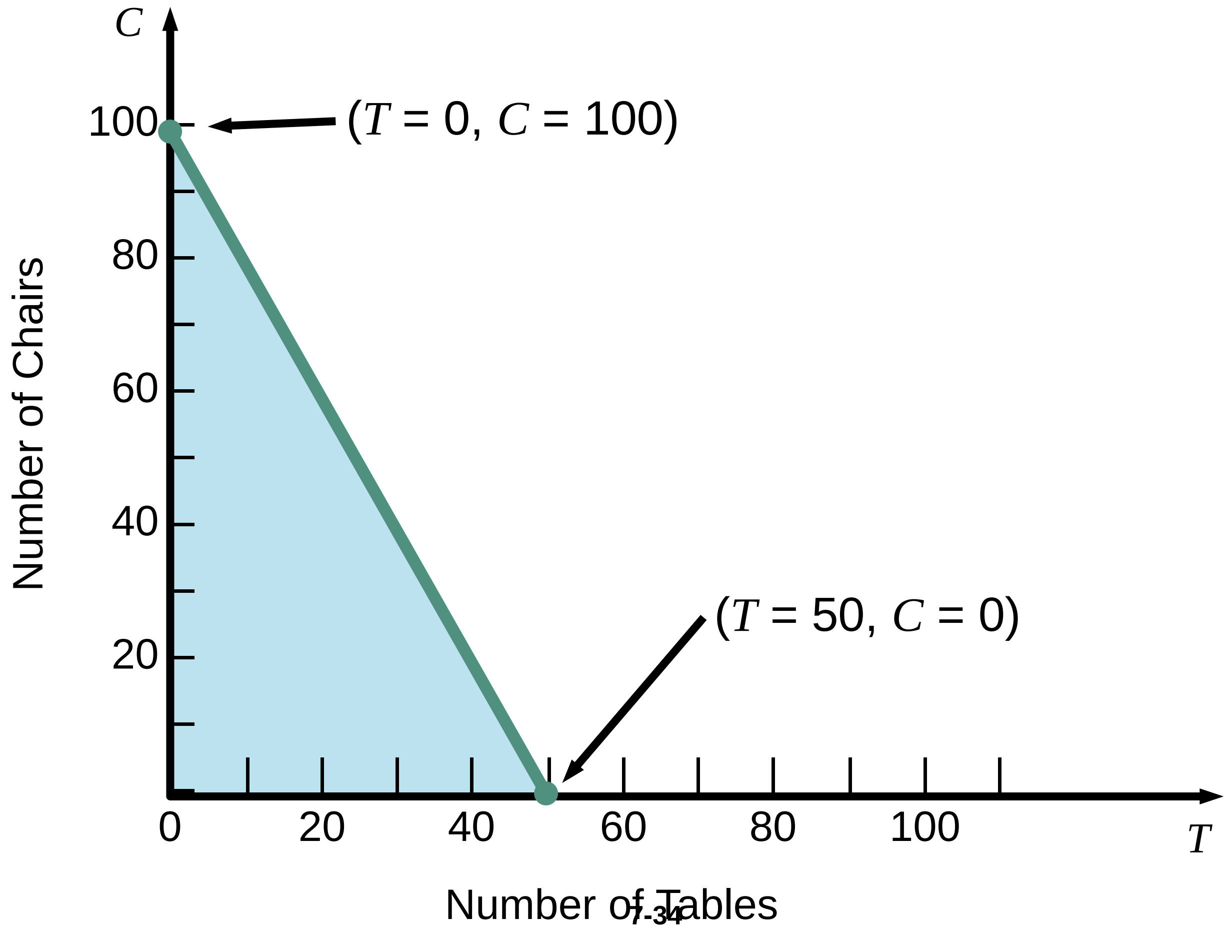
$$4(30) + 3(20) = 180$$

- The point (70, 40) lies above the line and does not satisfy the constraint

$$4(70) + 3(40) = 400$$

Graphical Representation of a Constraint (con't)

Region that Satisfies the Painting and Varnishing Constraint

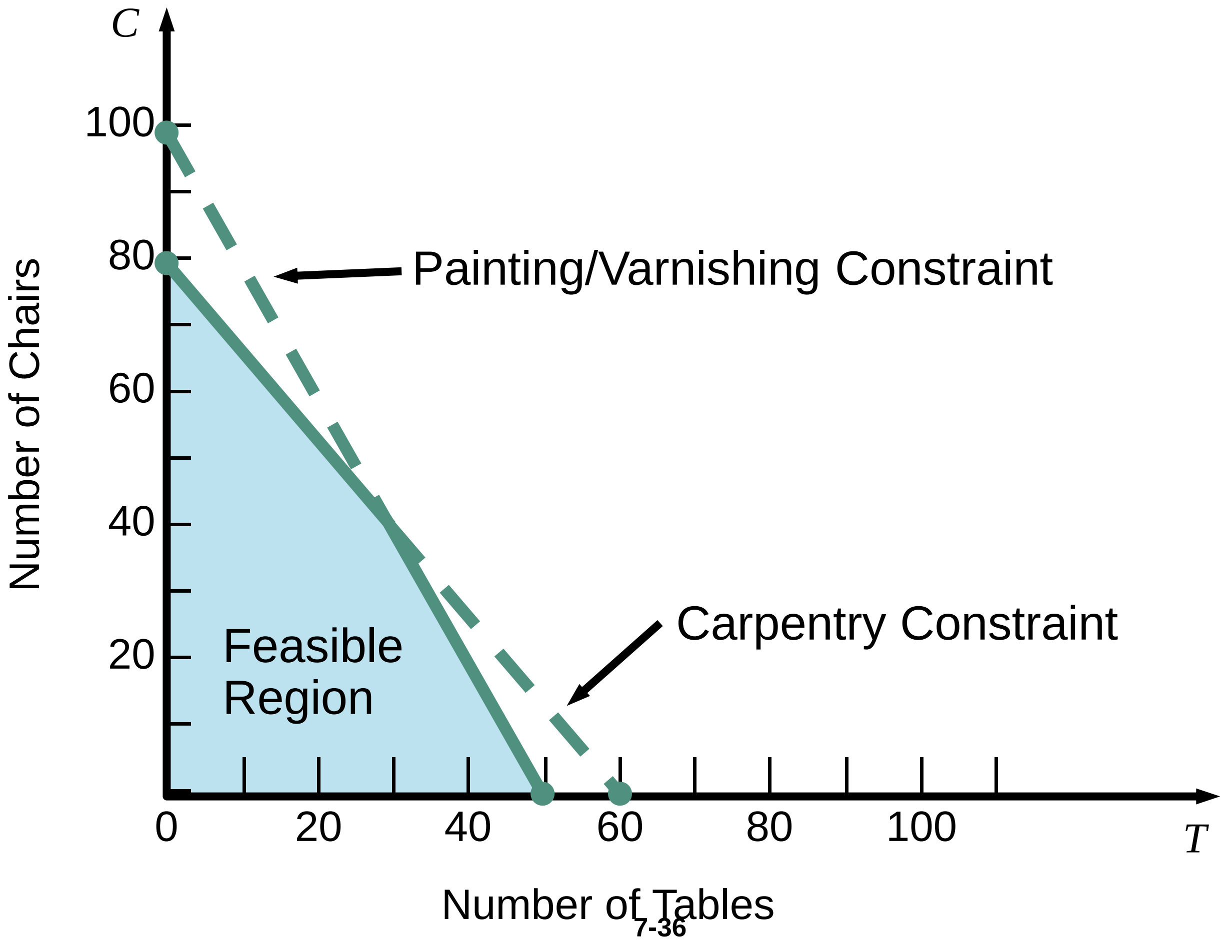


Graphical Representation of a Constraint (con't)

- To produce tables and chairs, both departments must be used
- We need to find a solution that satisfies both constraints *simultaneously*
- The following graph shows both constraint plots
- The *feasible region* (or *area of feasible solutions*) is where all constraints are satisfied
- Any point inside this region is a *feasible* solution
- Any point outside the region is an *infeasible* solution

Graphical Representation of a Constraint (con't)

Feasible Solution Region for the Furniture Problem



Graphical Representation of a Constraint

- For the point (30, 20)

Carpentry constraint	**4T + 3C ≤ 240 hours available** **(4)(30) + (3)(20) = 180 hours used**	✓
Painting constraint	**2T + 1C ≤ 100 hours available** **(2)(30) + (1)(20) = 80 hours used**	✓

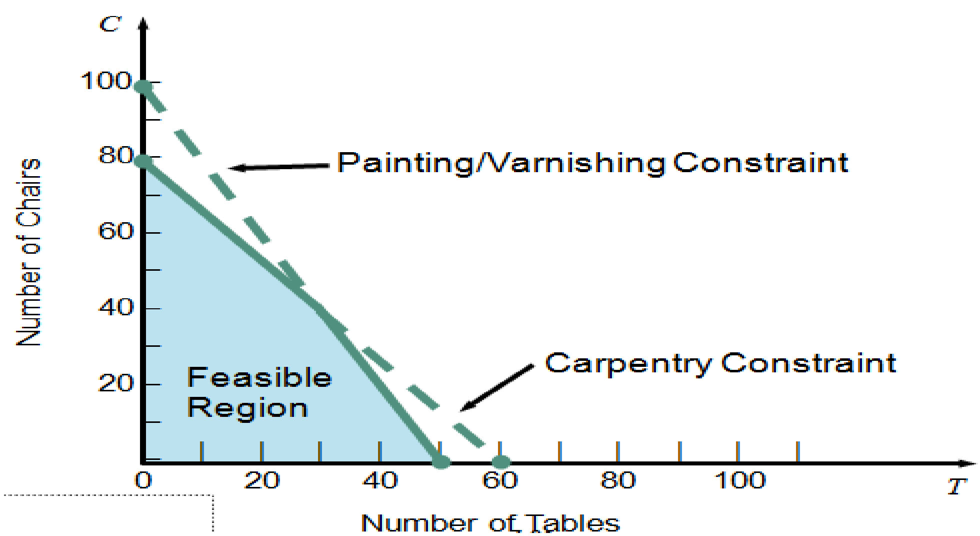
- For the point (70, 40)

Carpentry constraint	**4T + 3C ≤ 240 hours available** **(4)(70) + (3)(40) = 400 hours used**	✗
Painting constraint	**2T + 1C ≤ 100 hours available** **(2)(70) + (1)(40) = 180 hours used**	✗

Graphical Representation of a Constraint

- For the point (50, 5)

Carpentry constraint	**4*T* + 3*C* ≤ 240 hours available** **(4)(50) + (3)(5) = 215 hours used**	✓
Painting constraint	**2*T* + 1*C* ≤ 100 hours available** **(2)(50) + (1)(5) = 105 hours used**	✗



Isoprofit Line Solution Method

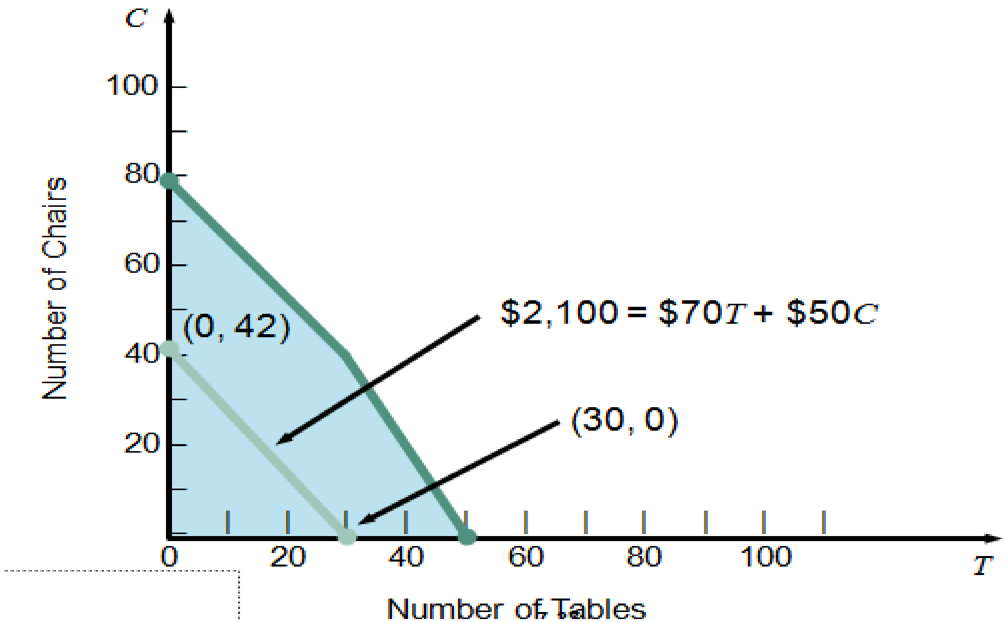
- Once the feasible region has been graphed, we need to find the optimal solution from the many possible solutions
- The speediest way to do this is to use the isoprofit line method
- Starting with a small but possible profit value, we graph the objective function
- We move the objective function line in the direction of increasing profit while maintaining the slope
- The last point it touches in the feasible region is the optimal solution

Isoprofit Line Solution Method

- Choose a profit of $2,100
- The objective function is then

$$\$2{,}100 = 70T + 50C$$

- Solving for the axis intercepts, we can draw the graph

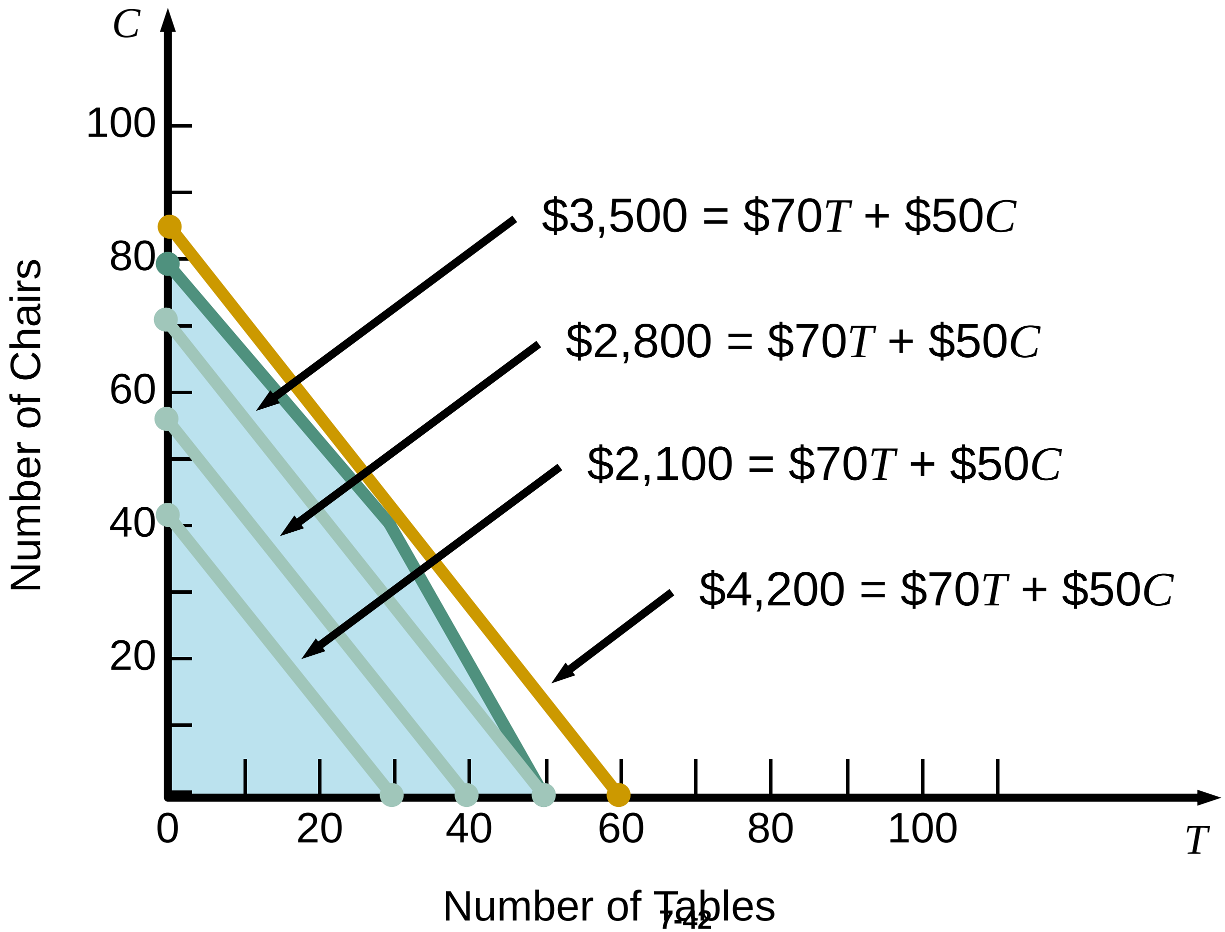


Isoprofit Line Solution Method (con't)

- This is obviously not the best possible solution
- Further graphs can be created using larger profits
- The further we move from the origin, the larger the profit will be
- The highest profit ($4,100) will be generated when the isoprofit line passes through the point (30, 40)

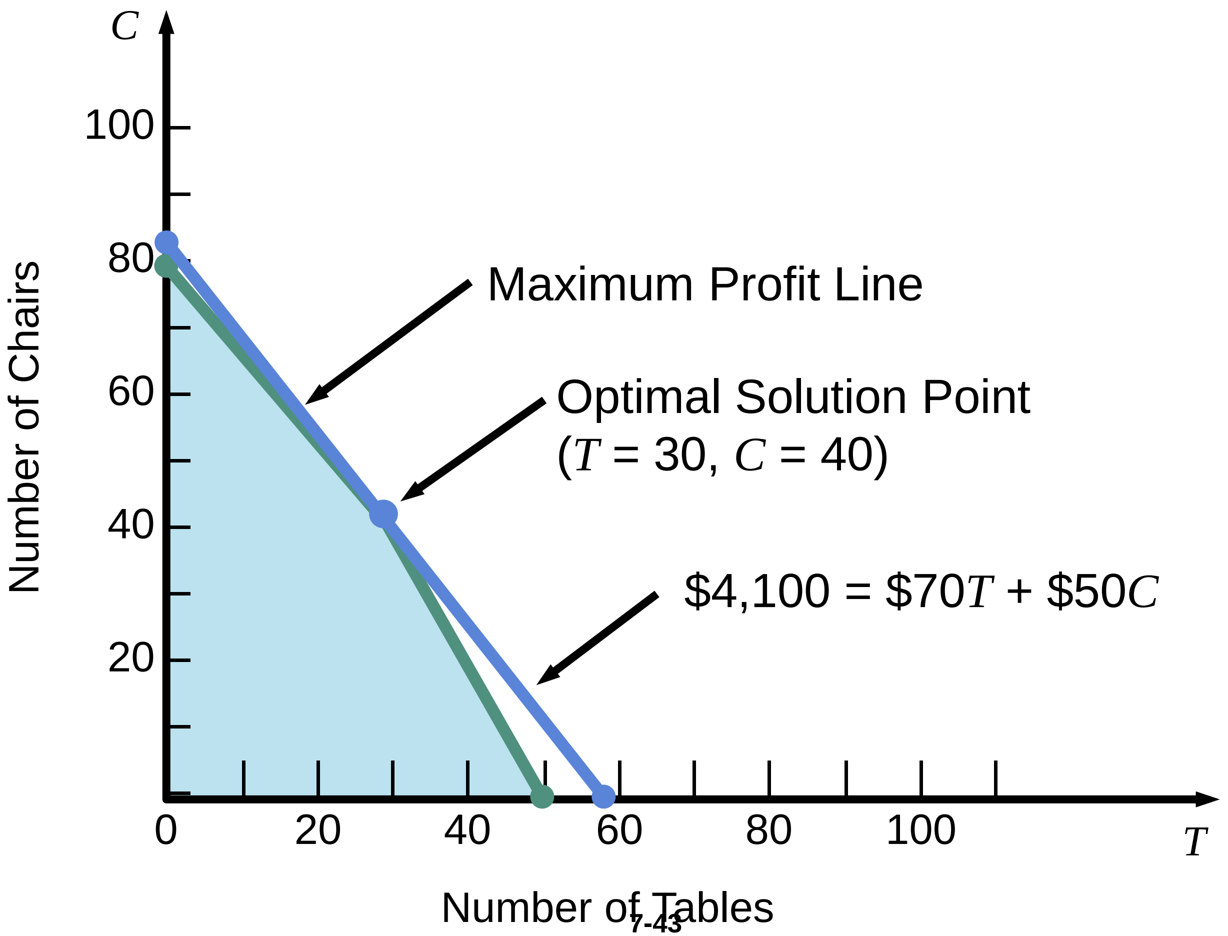
Isoprofit Line Solution Method (con't)

Four Isoprofit Lines Plotted for the Furniture Problem



Isoprofit Line Solution Method (con't)

Optimal Solution to the funiture problem

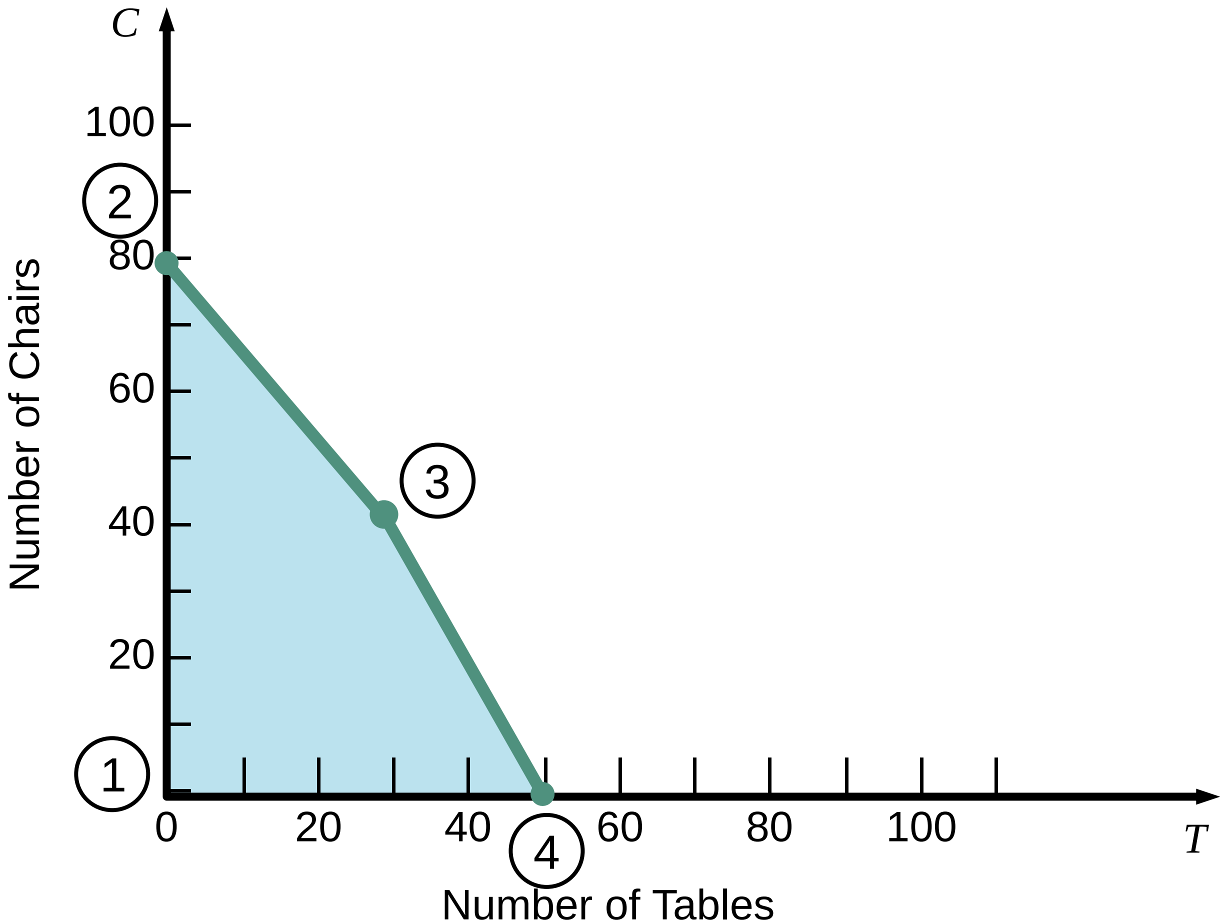


Corner Point Solution Method

- A second approach to solving LP problems employs the *corner point method*
- It involves looking at the profit at every corner point of the feasible region
- The mathematical theory behind LP is that the optimal solution must lie at one of the *corner points*, or *extreme point*, in the feasible region
- Here, the feasible region is a four-sided polygon with four corner points labeled 1, 2, 3, and 4 on the graph

Corner Point Solution Method (con't)

Four Corner Points of the Feasible Region



Corner Point Solution Method

- To find the coordinates for Point 3 accurately we have to solve for the intersection of the two constraint lines
- Using the *simultaneous equations method*, we multiply the painting equation by –2 and add it to the carpentry equation:

$$
\begin{aligned}
4T + 3C &= 240 && \text{(carpentry line)} \\
-4T - 2C &= -200 && \text{(painting line)} \\
C &= 40
\end{aligned}
$$

- Substituting 40 for C in either of the original equations allows us to determine the value of T:

$$
\begin{aligned}
4T + (3)(40) &= 240 && \text{(carpentry line)} \\
4T + 120 &= 240 \\
T &= 30
\end{aligned}
$$

Corner Point Solution Method (con't)

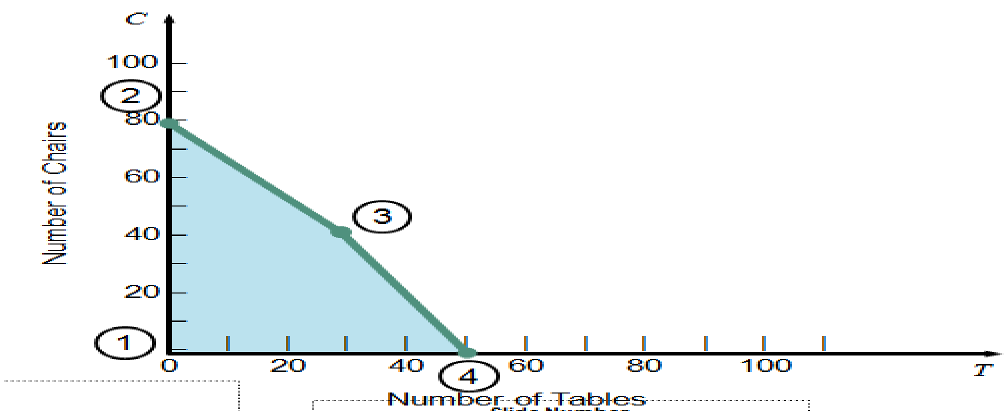
Point ① ($T = 0, C = 0$) Profit = \$70(0) + \$50(0) = \$0

Point ② ($T = 0, C = 80$) Profit = \$70(0) + \$50(80) = \$4,000

Point ④ ($T = 50, C = 0$) Profit = \$70(50) + \$50(0) = \$3,500

Point ③ ($T = 30, C = 40$) Profit = \$70(30) + \$50(40) = \$4,100

Because Point 3 returns the highest profit, this is the optimal solution



Slack and Surplus

- Slack is the amount of a resource that is not used
- For a less-than-or-equal constraint:
  - Slack = Amount of resource available – amount of resource used
- Surplus is used with a greater-than-or-equal constraint to indicate the amount by which the right hand side of the constraint is exceeded
  - Surplus = Actual amount – minimum amount

Summary of Graphical Solution Methods

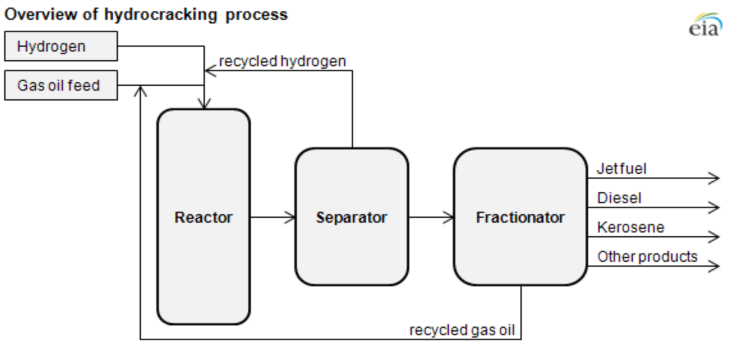
ISOPROFIT METHOD
1. Graph all constraints and find the feasible region.
2. Select a specific profit (or cost) line and graph it to find the slope.
3. Move the objective function line in the direction of increasing profit (or decreasing cost) while maintaining the slope. The last point it touches in the feasible region is the optimal solution.
4. Find the values of the decision variables at this last point and compute the profit (or cost).
CORNER POINT METHOD
1. Graph all constraints and find the feasible region.
2. Find the corner points of the feasible reason.
3. Compute the profit (or cost) at each of the feasible corner points.
4. Select the corner point with the best value of the objective function found in Step 3. This is the optimal solution.

Solving Minimization Problems

- Many LP problems involve minimizing an objective such as cost instead of maximizing a profit function
- Minimization problems can also be solved graphically by first setting up the feasible solution region and then using either the corner point method or an isocost line approach (which is analogous to the isoprofit approach in maximization problems) to find the values of the decision variables that yield the minimum cost

Generic Blending Example

- A raw material M can be made into either of two finished products A or B (for example crude oil can be made into gasoline or diesel fuel)
- The unit profit from using M in product A is $5
- The unit profit from using M in product B is $3
- Thus the profit z is (where x_1 is the amount of A and x_2 is the amount of B:
  - $z = 5x_1 + 3x_2$



Constraints

- If there were no limits on M (raw material), then we would use large amounts of M for both A and B
- If M were limited, then it would be better to use it all for product A
- However, each of the products A and B needs some other ingredients R, P, and Q (i.e. fuel additives)
- Product A requires that for each unit of M there be 4 lbs of P, 5 lbs of Q, and 3 lbs of R
- Product B requires that for each unit of M there be 5 lbs of P, 2 lbs of Q, and 8 lbs of R
- There are only 10 lbs/week available of R, 10 lbs/week of Q, and 12 lbs/week of R

Table of Limits

Ingredient	Lbs/unit of M in A	Lbs/unit of M in B	Lbs available per week
P	4	5	10
Q	5	2	10
R	3	8	12

Algebraic Formulation

[more complex problem with 3 constraints]

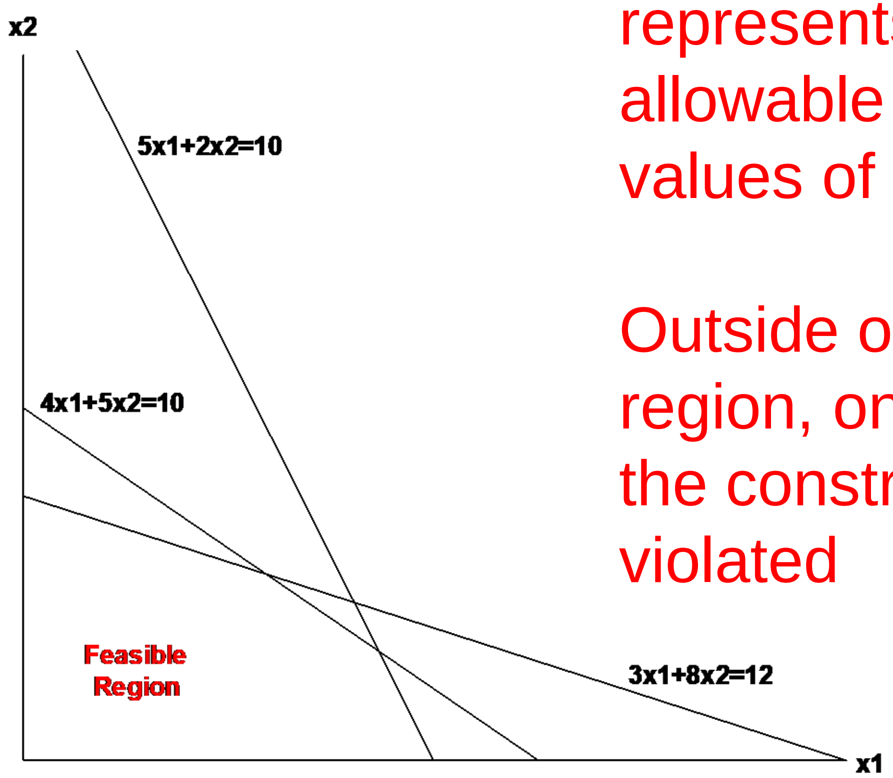
Ingredient	Lbs/unit of M in A	Lbs/unit of M in B	Lbs available per week
P	4	5	10
Q	5	2	10
R	3	8	12

- Objective function (to maximize):
  - $z = 5x_1 + 3x_2$
- Constraints:
  - $4x_1 + 5x_2 \leq 10$ (amount of P used)
  - $5x_1 + 2x_2 \leq 10$ (amount of Q used)
  - $3x_1 + 8x_2 \leq 12$ (amount of R used)
- Also in linear programming the variables (x_i) must be non-negative

Graphical View

- We can easily plot the constraints:
  - $4x_1 + 5x_2 \leq 10$
  - $5x_1 + 2x_2 \leq 10$
  - $3x_1 + 8x_2 \leq 12$
- The x_1 and x_2 intercepts are:
  - C1: When $x_1=0$, $x_2=2$; when $x_2=0$, $x_1=2.5$
  - C2: When $x_1=0$, $x_2=5$; when $x_2=0$, $x_1=2$
  - C3: When $x_1=0$, $x_2=1.5$; when $x_2=0$, $x_1=4$

Plot of Constraints



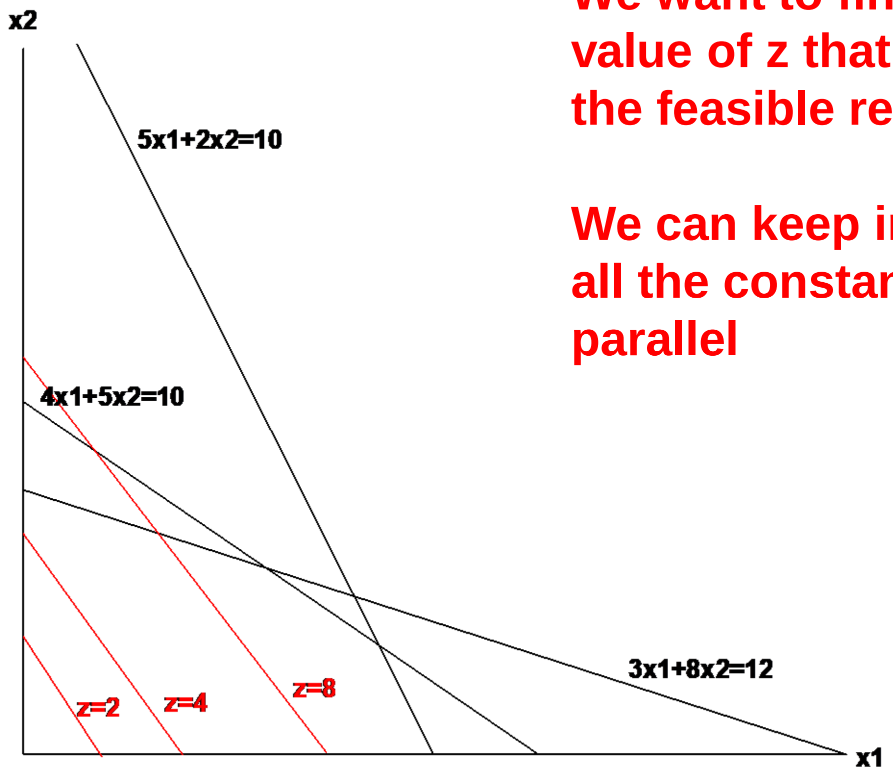
The "feasible region" represents the allowable values of x_1 and x_2

Outside of the feasible region, one or more of the constraints are violated

Plots of Objective Function

- We can also easily plot the objective function for various values of z:
  - For z=2
    - When x1=0 then x2=2/3
    - When x2=0 then x1=2/5
  - For z=4
    - When x1=0 then x2=4/3
    - When x2=0 then x1=4/5
- These are “isocost” or “isoprofit” lines

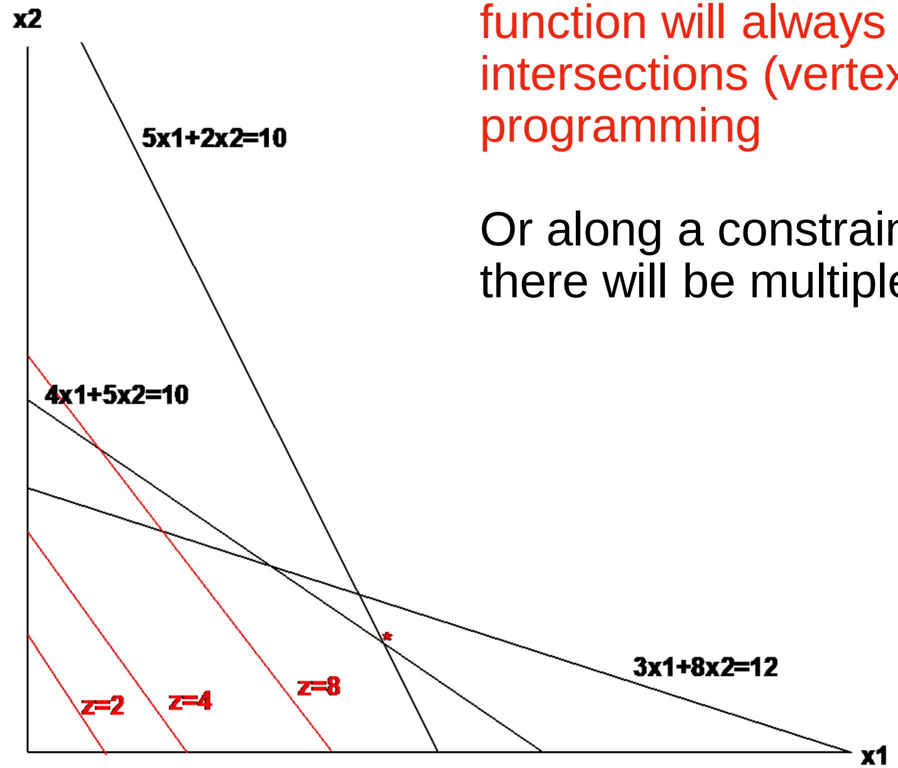
Plot of Objective Function



We want to find the largest value of z that is inside the feasible region

We can keep increasing z, as all the constant z lines are parallel

Graphical Solution Method (con't)



The optimum value of the objective function will always be on constraint intersections (vertex) in linear programming

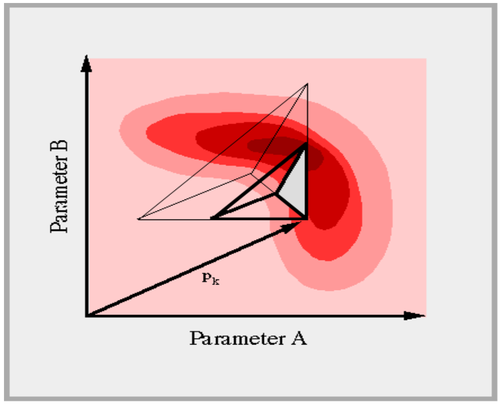
Or along a constraint, in which case there will be multiple solution values

Graphical Solution Method (con’t)

- In this case the largest z line that is in the feasible region is 180/17 or about 10.588
- Here the optimum is on the two constraints $5x_1+2x_2=10$ and $4x_1+5x_2=10$
- Solving these 2 equations in 2 unknowns also yields:
  - $X_1=30/17$ and $x_2=10/17$; yielding $z=180/17$

Four Special Cases in LP

- Four special cases and difficulties arise at times when solving LP problems:
  - Infeasibility
  - Unboundedness
  - Redundancy
  - Alternate Optimal Solutions

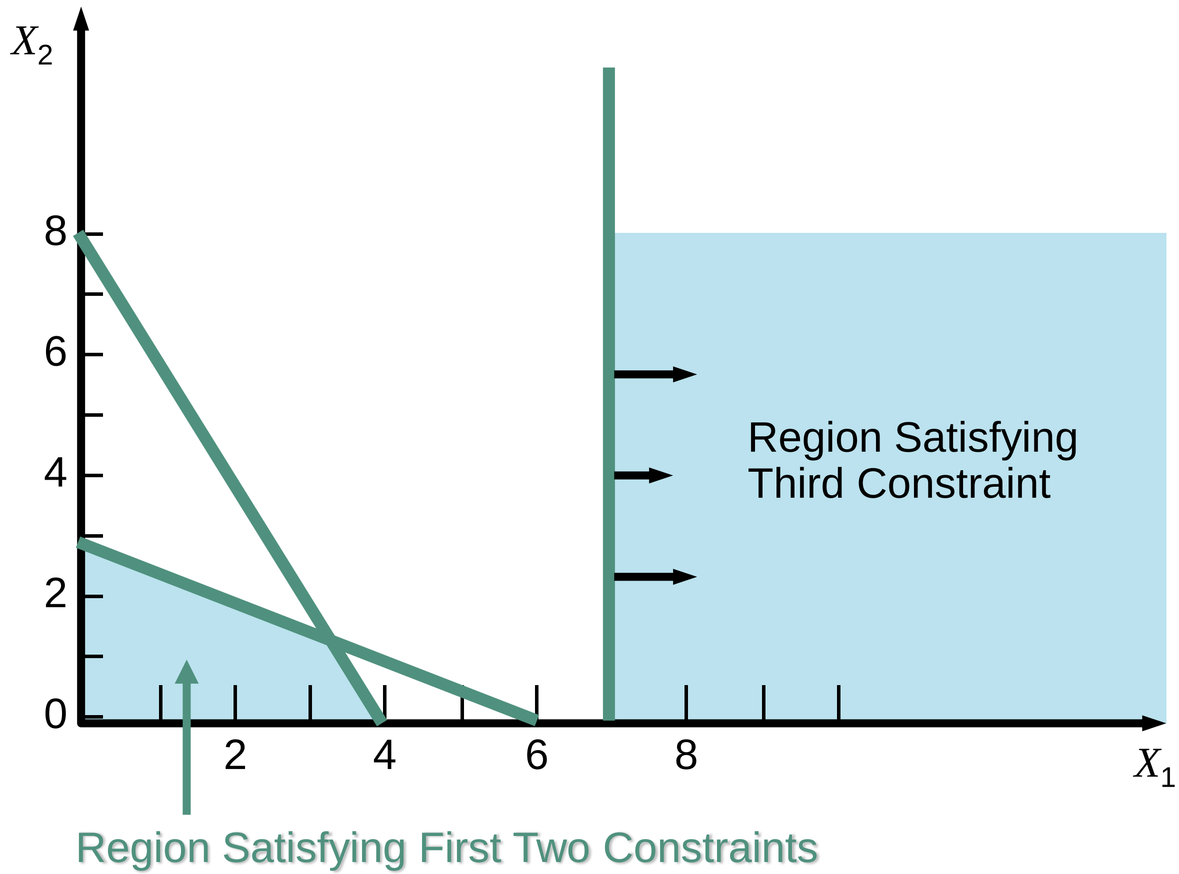


Infeasibility

- No feasible solution
  - Exists when there is no solution to the problem that satisfies all the constraint equations
  - No feasible solution region exists
  - This is a common occurrence in the real world
  - Generally one or more constraints are relaxed until a solution is found

Infeasibility (con't)

- A problem with no feasible solution

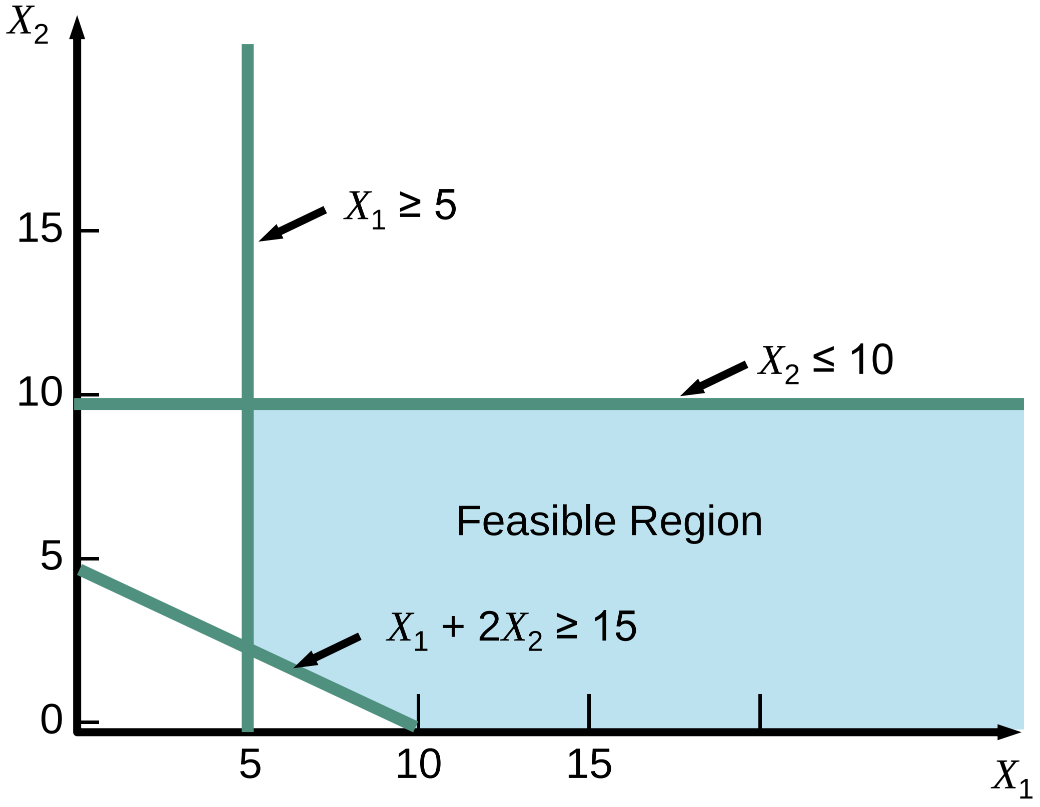


Unboundedness

- Sometimes a linear program will not have a finite solution
- In a maximization problem, one or more solution variables, and the profit, can be made infinitely large without violating any constraints
- In a graphical solution, the feasible region will be open ended
- This usually means the problem has been formulated improperly, or a constraint ommited

Unboundedness (con't)

- A solution region unbounded to the right

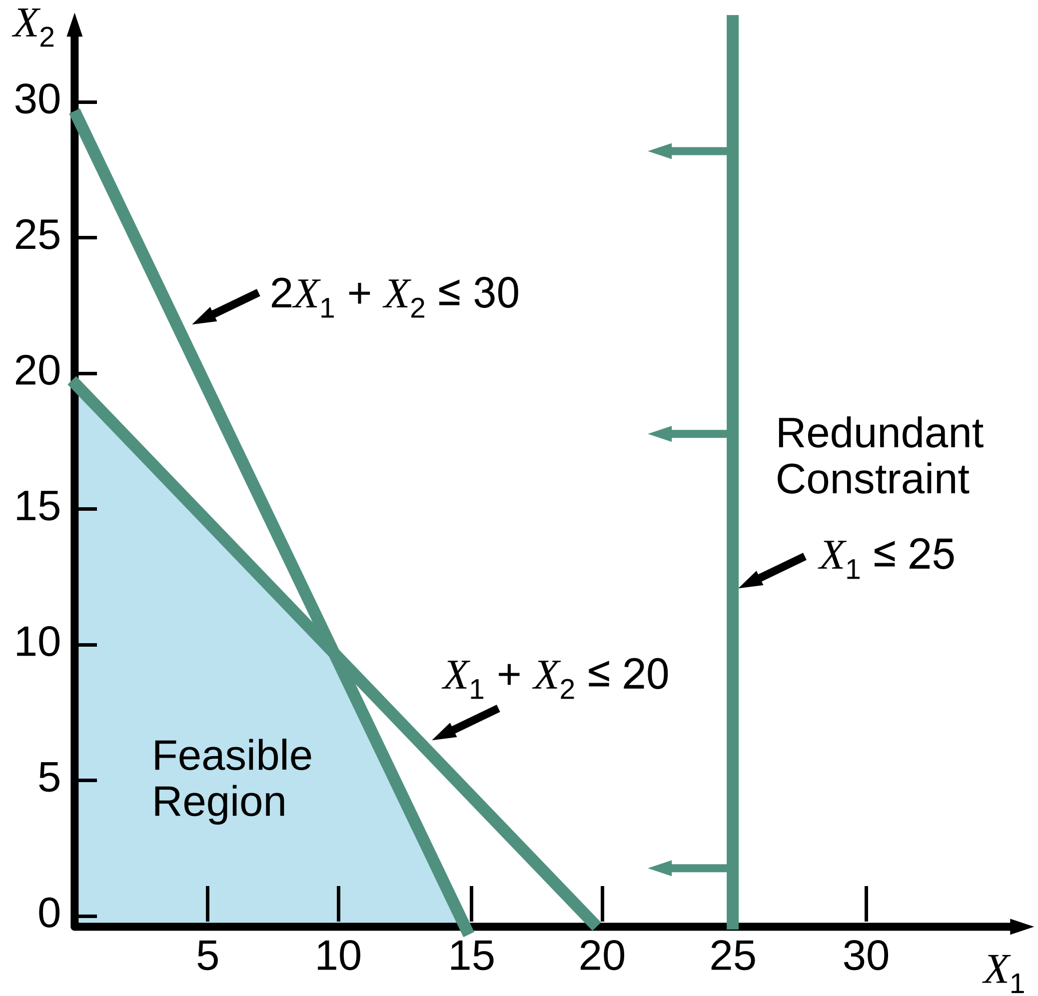


Redundancy

- A redundant constraint is one that does not affect the feasible solution region
- One or more constraints may be more binding
- This is a very common occurrence in the real world
- It causes no particular problems, but eliminating redundant constraints simplifies the model

Redundancy (con't)

- A problem with a redundant constraint

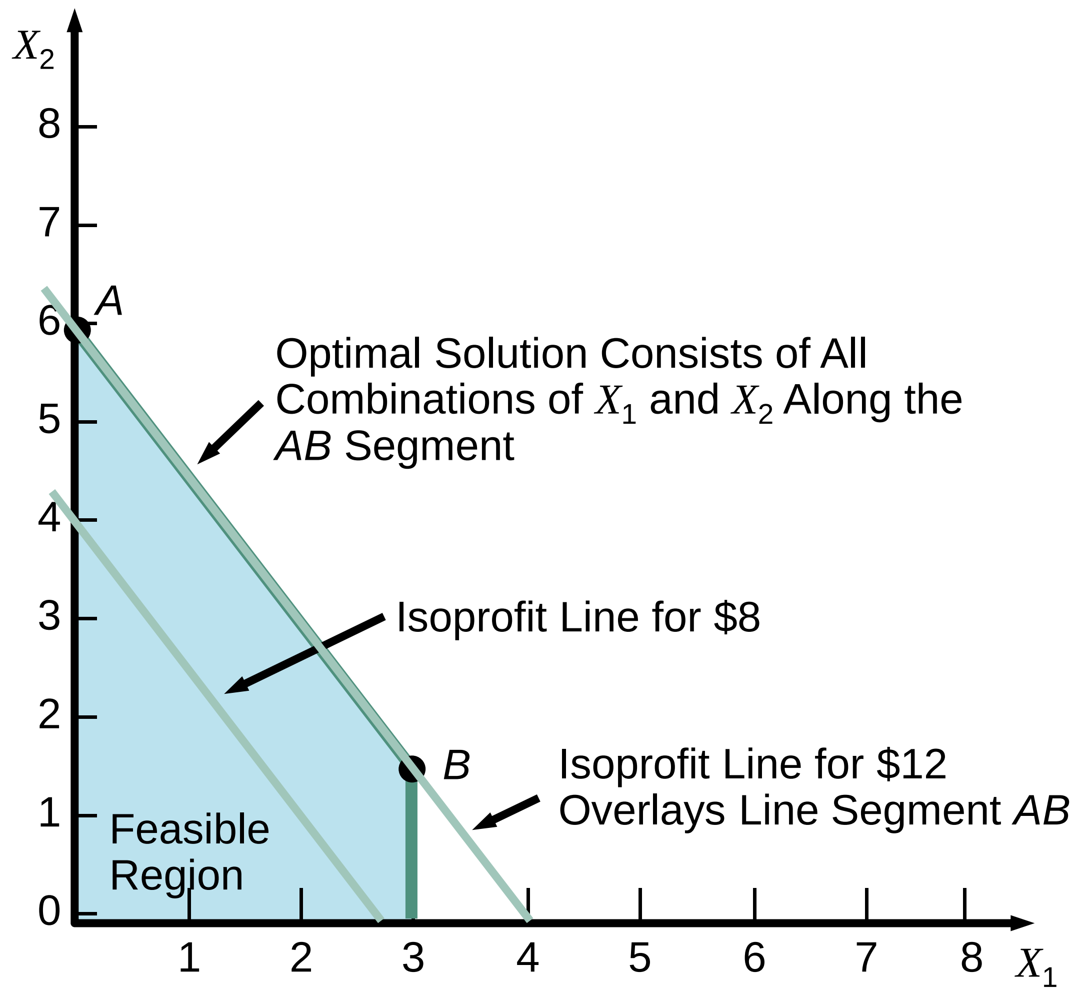


Alternate Optimal Solutions

- Occasionally two or more optimal solutions may exist
- Graphically this occurs when the objective function's isoprofit or isocost line runs perfectly parallel to one of the constraints
- This actually allows management great flexibility in deciding which combination to select as the profit is the same at each alternate solution

Alternate Optimal Solutions (con’t)

- Example of alternate optimal solutions



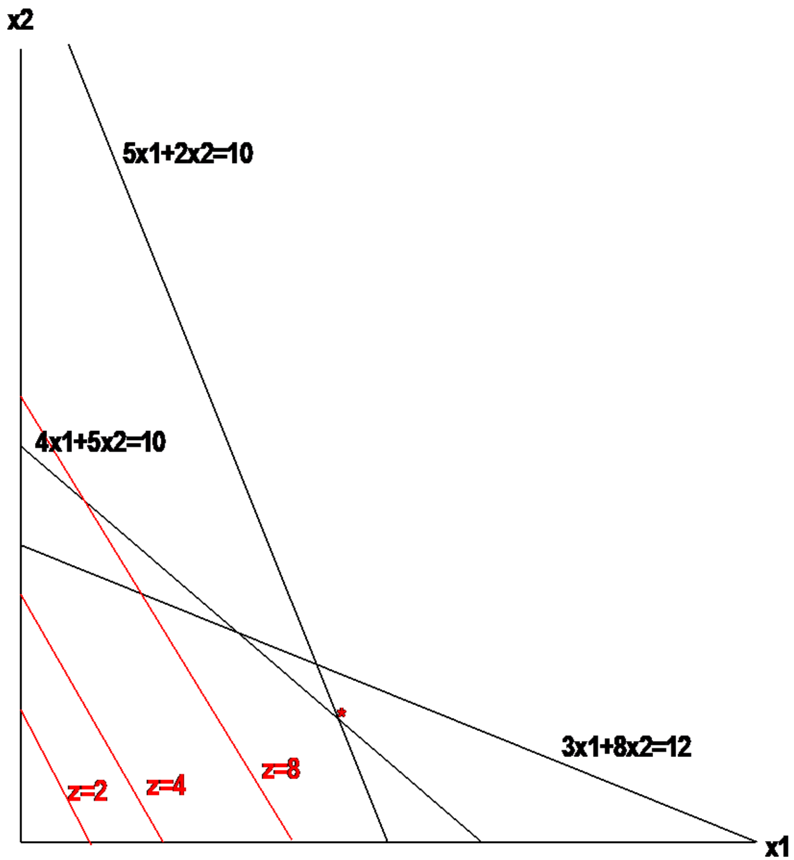
Computer Techniques

- When there are only two variables, a graphical solution is possible
- For more than two variables, computer techniques must be used
- Also when there are many constraints, even with only two variables, graphical methods become cumbersome

Computer Techniques (con't)

- Computer linear programming methods were first developed by Leonid Kantorovich in 1939
- He developed these for use during World War II to plan expenditures and returns (battle/encounter locations) in order to reduce costs to the army and increase losses to the enemy
- The methods was kept secret until 1947 when George Dantzig published the simplex method and John von Neumann developed the theory of duality

Higher Dimensional Problesm



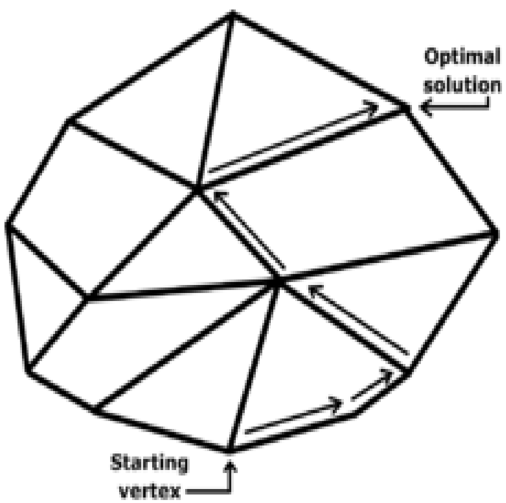
The optimum value of the objective function will always be on constraint intersections (vertex) in linear programming (if not a special situation)

For two dimensions, one just has to solve two equations in two unknowns for each vertex

For n-dimensional problems, one must solve n equations in n unknowns for each vertex – <u>the trick is to find an efficient method for examining vertices</u>

Simplex Method

- The first really successful method to solve the linear programming problem was the simplex method by Danzig (“Activity Analysis of Production and Allocation,Cowles Commission Monograph 13, John Wiley & Sons, 1951)
- The simplex method systematically examines the vertices
- Today there are several successful methods, some derived from the simplex method

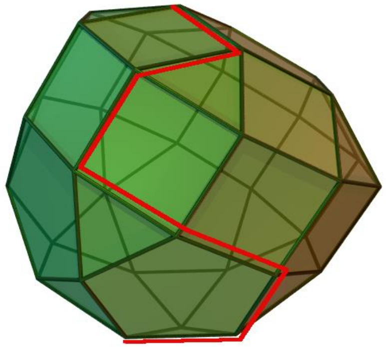


Slack Variables

- The Simplex Method introduces positive “slack variables” for each constraint to turn the inequality into an equality:
  - $4x_1 + 5x_2 + x_3 = 10$
  - $5x_1 + 2x_2 + x_4 = 10$
  - $3x_1 + 8x_2 + x_5 = 12$
- <u>The value of a slack variable is a measure of how far the optimum point is away from the constraint</u>

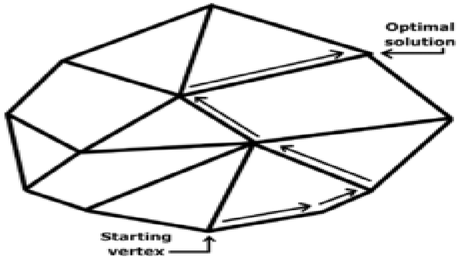
Simplex Order

- If there are N variables and M constraints, there will be M slack variables for a total of (M+N) total variables
- The simplex methods systematically finds the vertex upon which the objective function is maximized (the vertex will have N of the M equations with zero for the slack variables)
- The order of the simplex method is M^3



Simplex Algorithm

[textbook online appendices]



- 1. Chose any M of the variables to have positive values and the other N to be zero; those chosen are the "basic" variables. In the first step, one normally picks the slack variables to be in the first basis, and the others to be zero (start at origin).
- 2. Eliminate the basic variables from the objective function by solving M equations in M unknowns, so that the objective function and constrain equations are expressed in terms of the non-basic variables.
- 3. Find out how much each non-basic variable can be increased without violating any constraint. If the coefficients of the non-basic variables in the objective function are not positive, then none can be increased and a solution has been found. The non-basic variable that increases the objective function the most (by the largest possible increase in that non-basic variable) is chosen to become one of the basic variables.
- 4. Replace the one of the current basic variables which becomes zero when the chosen variable is increased to its largest permissible value. This step is called "pivoting".
- 5. Go back to step 2.

Iteration 1

(x3, x4, x5 in basis; x1 and x2 are zero; start at origin)

- Objective Function in terms of non-basic variables:
  - $0 = z - 5x_1 - 3x_2$ (x1 has most effect on increasing z)
- Constraints in terms of non-basis variables:
  - $10 = 4x_1 + 5x_2 + x_3$
  - $10 = 5x_1 + 2x_2 + x_4$ (if x1 is increased this constraint will hit first)
  - $12 = 3x_1 + 8x_2 + x_5$
- Three equations in 3 unknowns (x3,x4,x5), since x1 and x2 are zero

Iteration 1 (in table form)

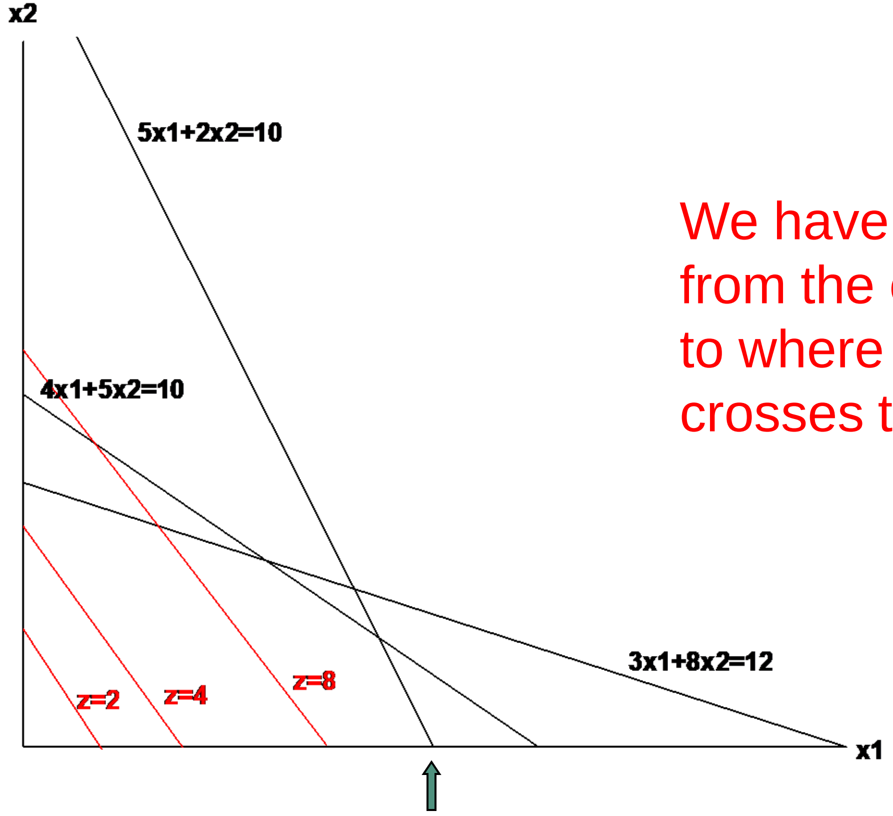
[x1 will join the basis (since -5 is largest); x4 will leave the basis since 10/5 is smaller than 10/4 or 12/3; “pivot” in red]

Basis	Value	X1	X2	X3	X4	x5
Z	0	-5	-3			
X3	10	4	5	1		
X4	10	5	2		1	
X5	12	3	8			1

Iteration 1 (con't)
(x3, x4, x5 in basis; x1 and x2 are zero)

- Objective Function in terms of non-basic variables:
  - $0 = z - 5x_1 - 3x_2$ (x1 has most effect on increasing z; x1 joins basis)
- Constraints in terms of non-basis variables:
  - $10 = 4x_1 + 5x_2 + x_3$
  - $10 = 5x_1 + 2x_2 + x_4$ (if x1 is increased this constraint will hit first)
  - $12 = 3x_1 + 8x_2 + x_5$
- Solve for x1 in above constraint:
  - $X_1 = (10 - 2x_2 - x_4)/5$
- And substitute for x1 in other 3 equations (to get rid of x1)

Iteration 1



We have now moved from the origin (x1=x2=0) to where the x4 constraint crosses the x1 axis.

Iteration 2 (x3, x1, x5 in basis)

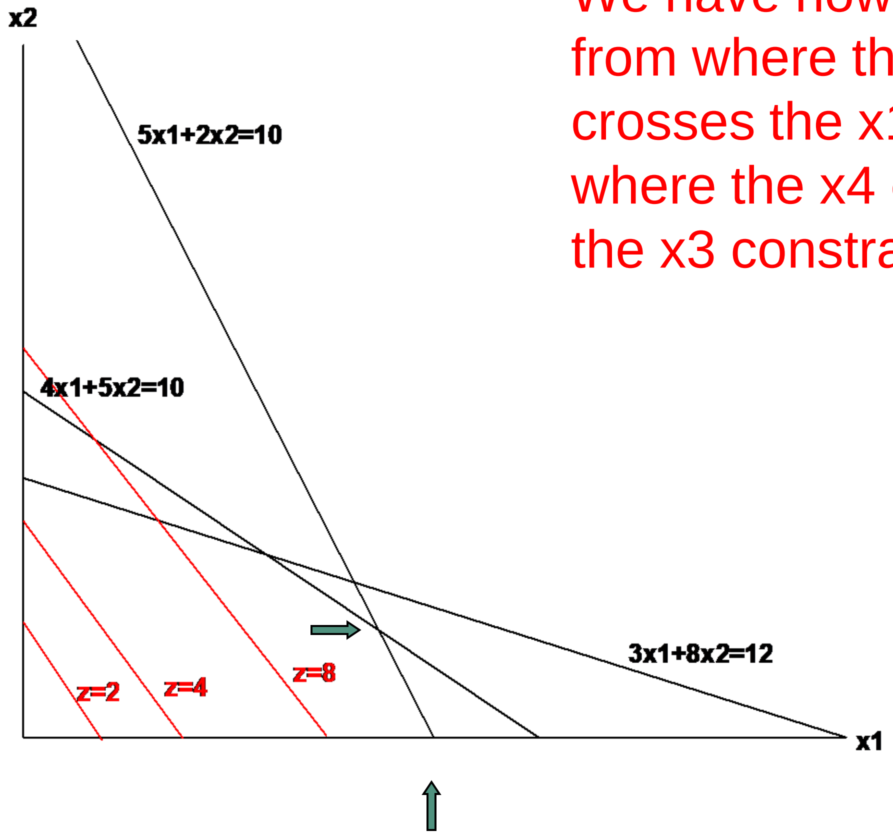
- Objective Function in terms of non-basic variables:
  - $10 = z - x_2 + x_4$
- Constraints in terms of non-basis variables:
  - $2 = (17/5)x_2 - (4/5)x_4 + x_3$
  - $2 = (2/5)x_2 + (1/5)x_4 + x_1$
  - $6 = (34/5)x_2 - (4/5)x_4 + x_5$

Iteration 2 (in table form)

[x2 will join the basis (since -1 is largest); x3 will leave the basis since 2/(17/5) is smallest; "pivot" in red]

Basis	Value	X1	X2	X3	X4	x5
Z	10	0	-1	0	1	0
X3	2	0	17/5	1	-4/5	0
X1	2	1	2/5	0	1/5	0
X5	6	0	34/5	0	-4/5	1

Iteration 2



We have now moved
from where the x4 constraint
crosses the x1 axis to
where the x4 constraint crosses
the x3 constraint.

Iteration 3 (in table form)

[no further improvement in z]

Basis	**Value**	**X1**	**X2**	**X3**	**X4**	**x5**
Z	180/17	0	0	5/17	3/17	0
X2	10/17	0	1	5/17	-4/17	0
X1	30/17	1	0	-2/17	5/17	0
X5	2	0	0	-2	1	1

Duality

- The duality theorem states there is an alternative way of formally stating an LP problem which may be easier (or more insightful) or faster than the original formulation
- The duality concept exchanges the variables and the constraints
- Since the time to solve an LP problem is largely dependent on the number of constraints, the dual version may be faster to solve

Our Example in Duality

- Min $w = 10w_1 + 10w_2 + 12\ w_3$
- Subject to:
  - $4w_1 + 5w_2 + 3w_3 \geq 5$
  - $5w_1 + 2w_2 + 8w_3 \geq 3$

- Original problem:
  - Objective function:
    - Max $z = 5x_1 + 3x_2$
  - Subject to:
  - $4x_1 + 5x_2 \leq 10 \quad 5x_1 + 2x_2 \leq 10 \quad 3x_1 + 8x_2 \leq 12$

Matrix Notation

- Primal:
  - Max z = CX
  - Subject to: AX ≤ D and X ≥ 0
- Dual:
  - Min w = WD
  - Subject to: WA ≥ C and W ≥ 0
- Where X,C,D,W are vectors and A is a matrix

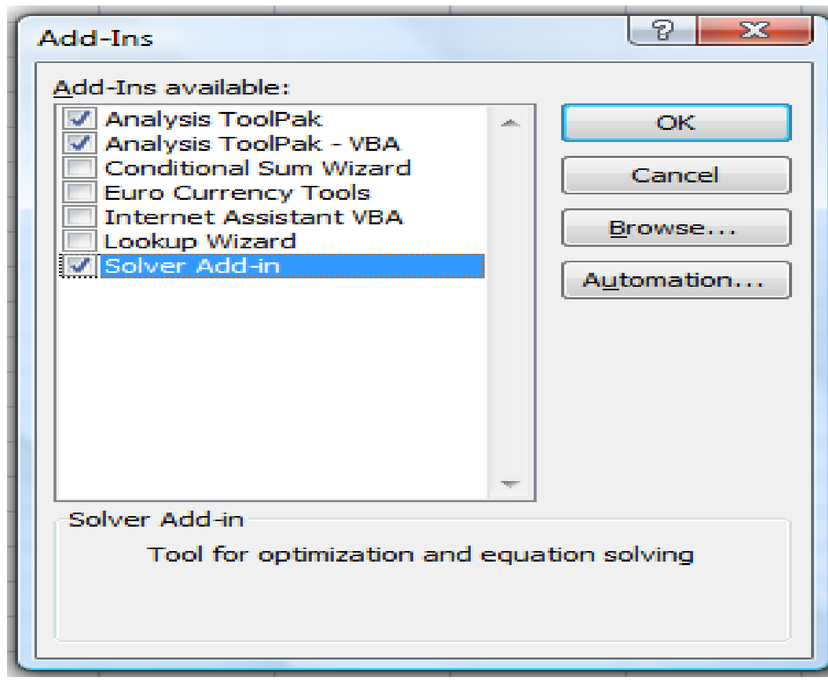
Matrix Solution

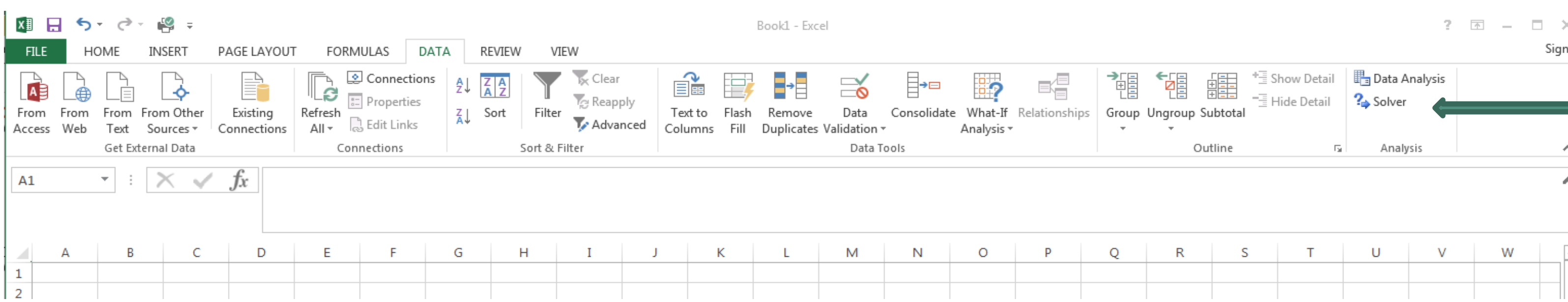
- Computer implementation of the simplex (and related) methods uses matrix algebra
- A major subset of the algorithms is solving a set of linear simultaneous equations, and decomposition methods are used instead of taking matrix inversion
- The most efficient algorithms use “sparse matrix” techniques, where only non-zero elements in the matrices are stored and operated upon – Excel and QM are not in that category
- Many commercial applications involve hundreds of variables and/or constraints, and thus much more rigorous software must be used

Linear Programming in Excel

- Make sure “Solver” is enabled in the Add-ins
  - Solver is limited to 200 variables and 100 constraints
- Set up raw data in spreadsheet
- Select Solver
- Specify objective function and criteria
- Specify independent variables
- Specify constraints (including that independent variables be positive)
- Select “Solve”

Solver Add-in





Blending Example in Excel

[column A is just “labels”]

Microsoft Excel - Book2

File Edit View Inse

E3

- Objective function:
  - Max $z = 5x_1 + 3x_2$
- Subject to:
- $4x_1 + 5x_2 \le 10 \quad 5x_1 + 2x_2 \le 10 \quad 3x_1 + 8x_2 \le 12$

	A	B	C
1	Variables	Formulas	Constraint Values
2	x1	0	
3	x2	0	
4	z	=5*B2+3*B3	
5	x3	=4*B2+5*B3	10
6	x4	=5*B2+2*B3	10
7	x5	=3*B2+8*B3	12
8			

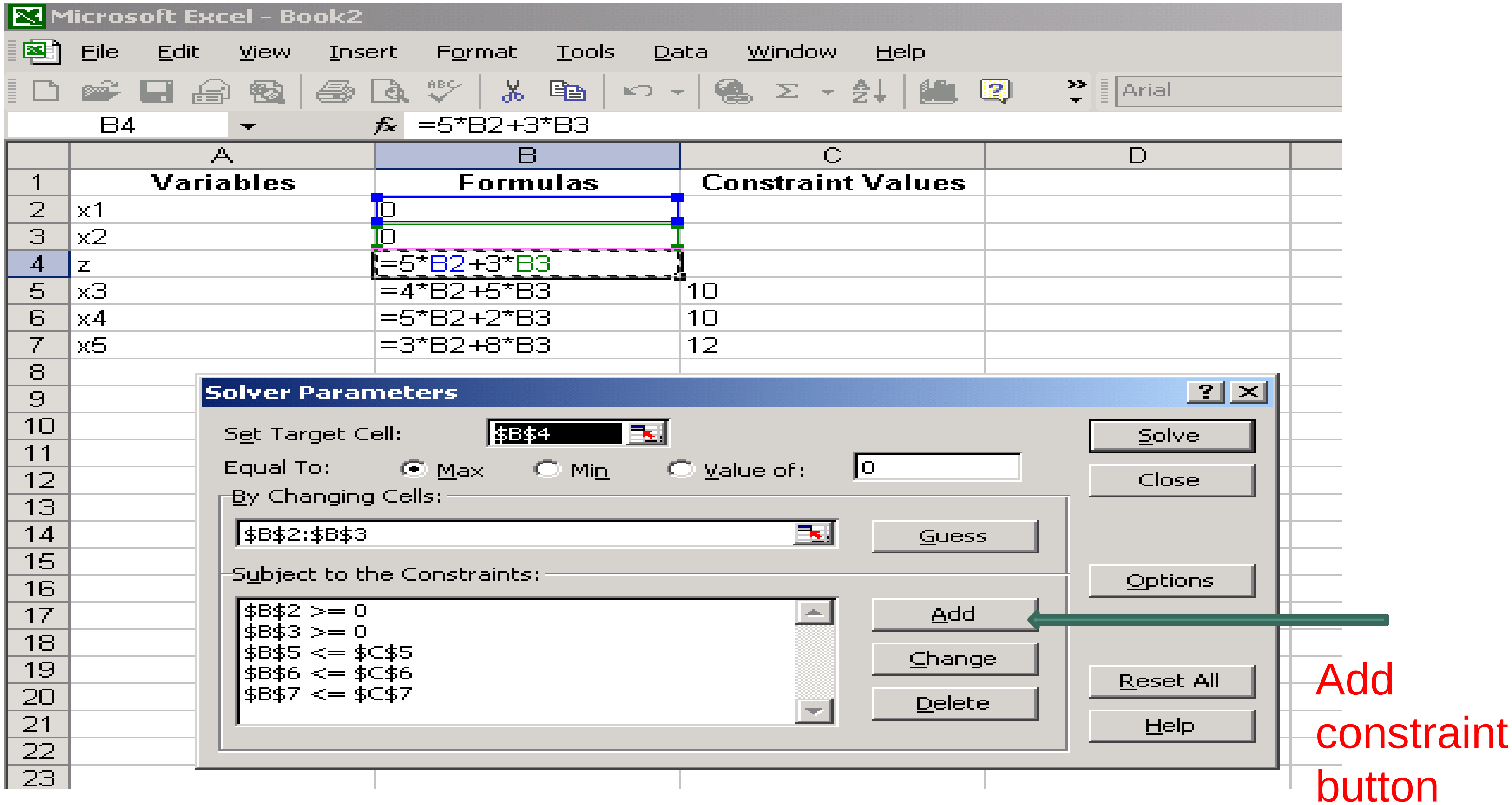
Objective function

Constraints

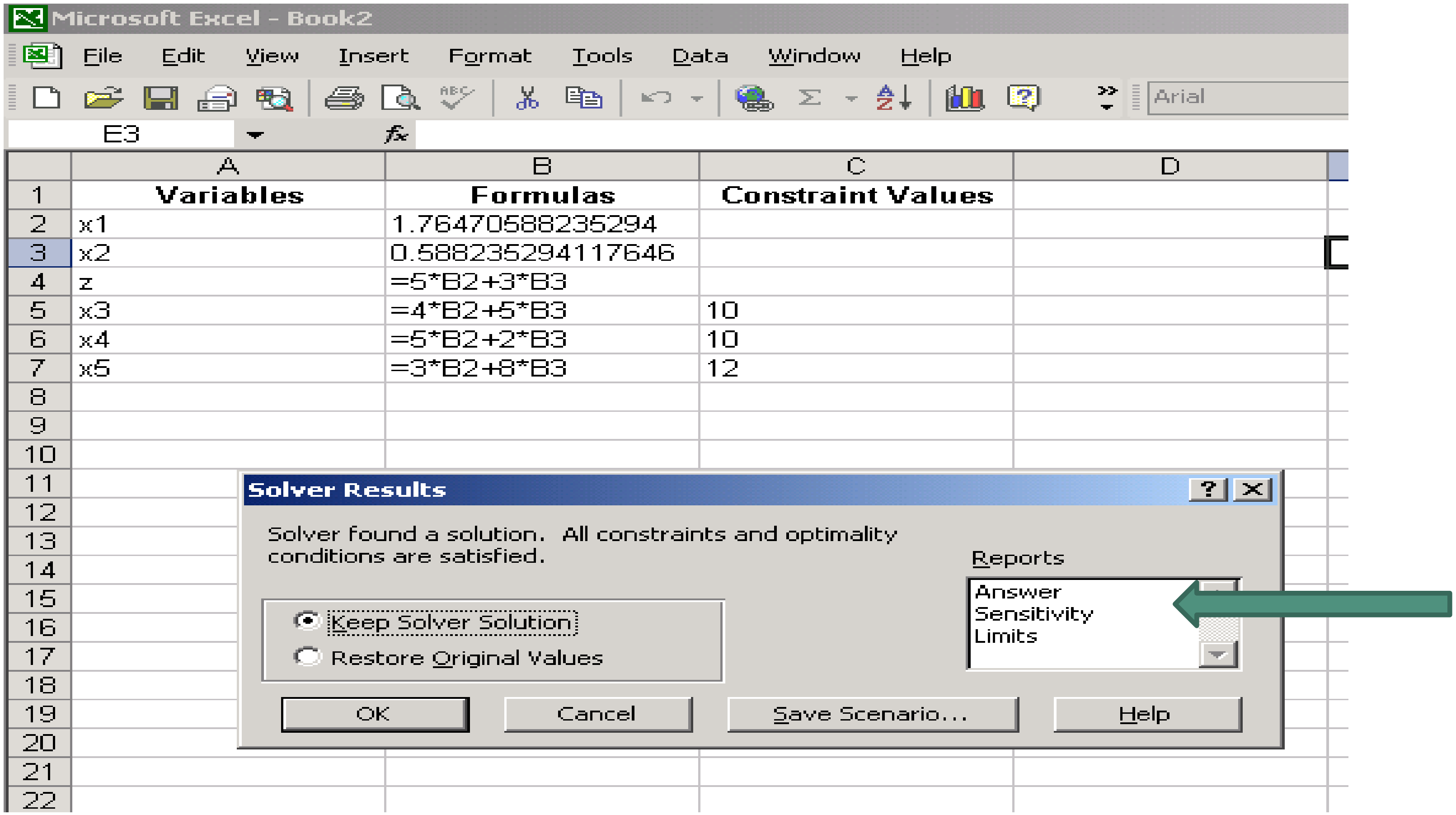
Slack variables

After invoking solver in Excel 2007…

[more options in newer Excel versions, plus check-box for non-negative variables]

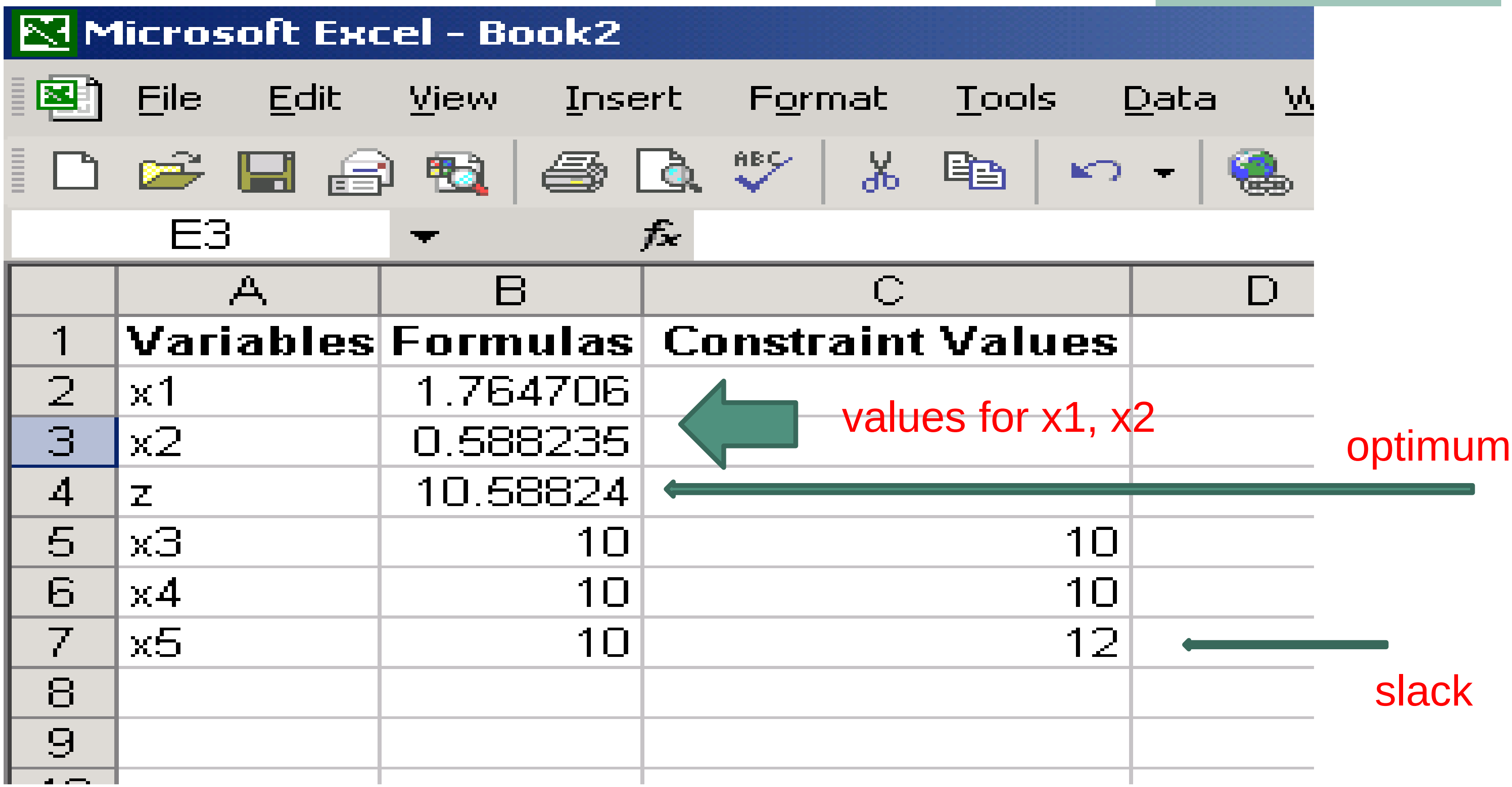


After selecting “solve” …



After picking "answer"

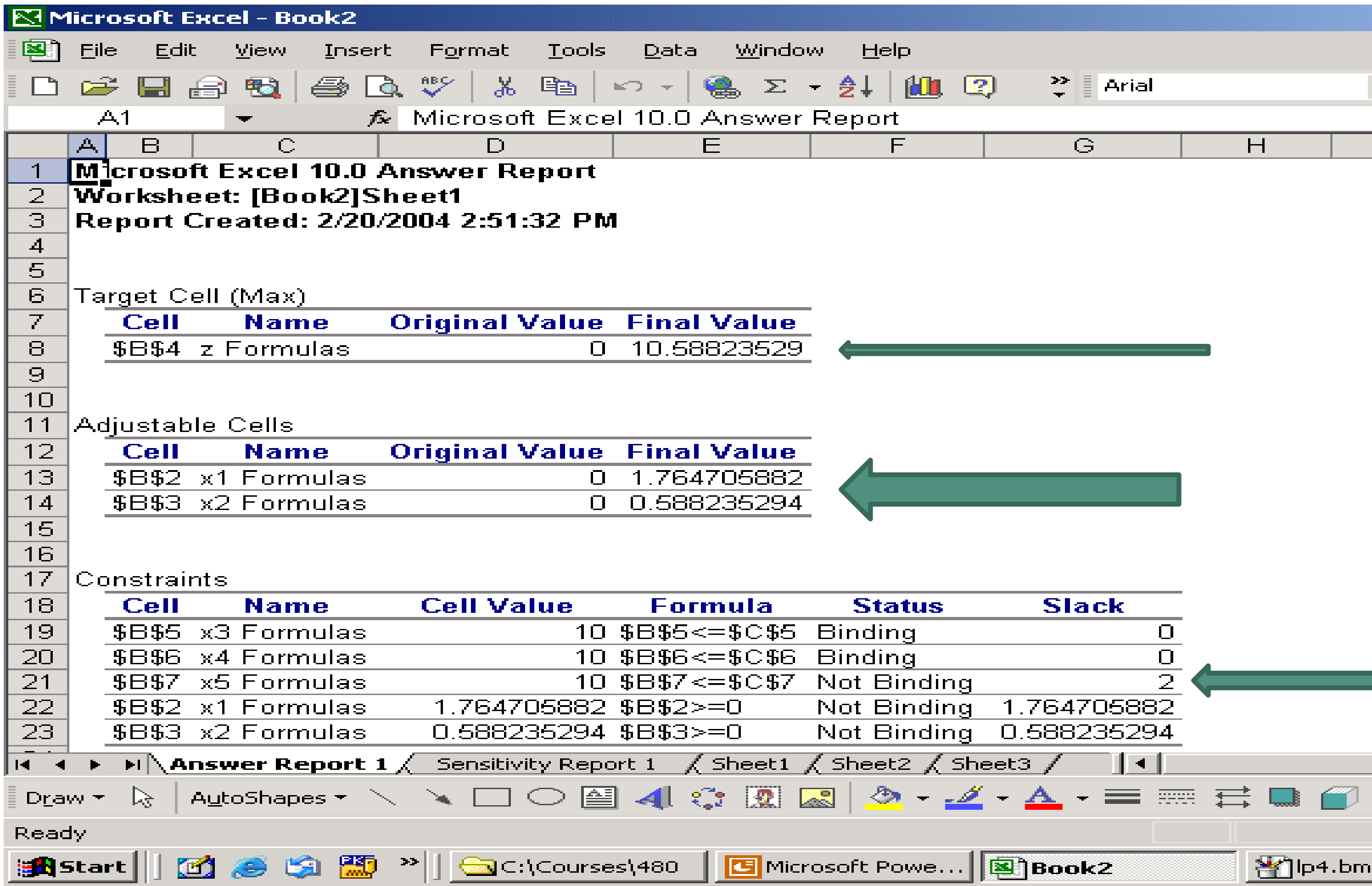
[and viewing data instead of formulas]



	A	B	C	D
1	Variables	Formulas	Constraint Values	
2	x1	1.764706		
3	x2	0.588235		
4	z	10.58824		
5	x3	10	10	
6	x4	10	10	
7	x5	10	12	
8				
9				

Generated "Answer Report"

[built in separate tab (worksheet), also shows "slack"]



Microsoft Excel 10.0 Answer Report
Worksheet: [Book2]Sheet1
Report Created: 2/20/2004 2:51:32 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$B$4	z Formulas	0	10.58823529

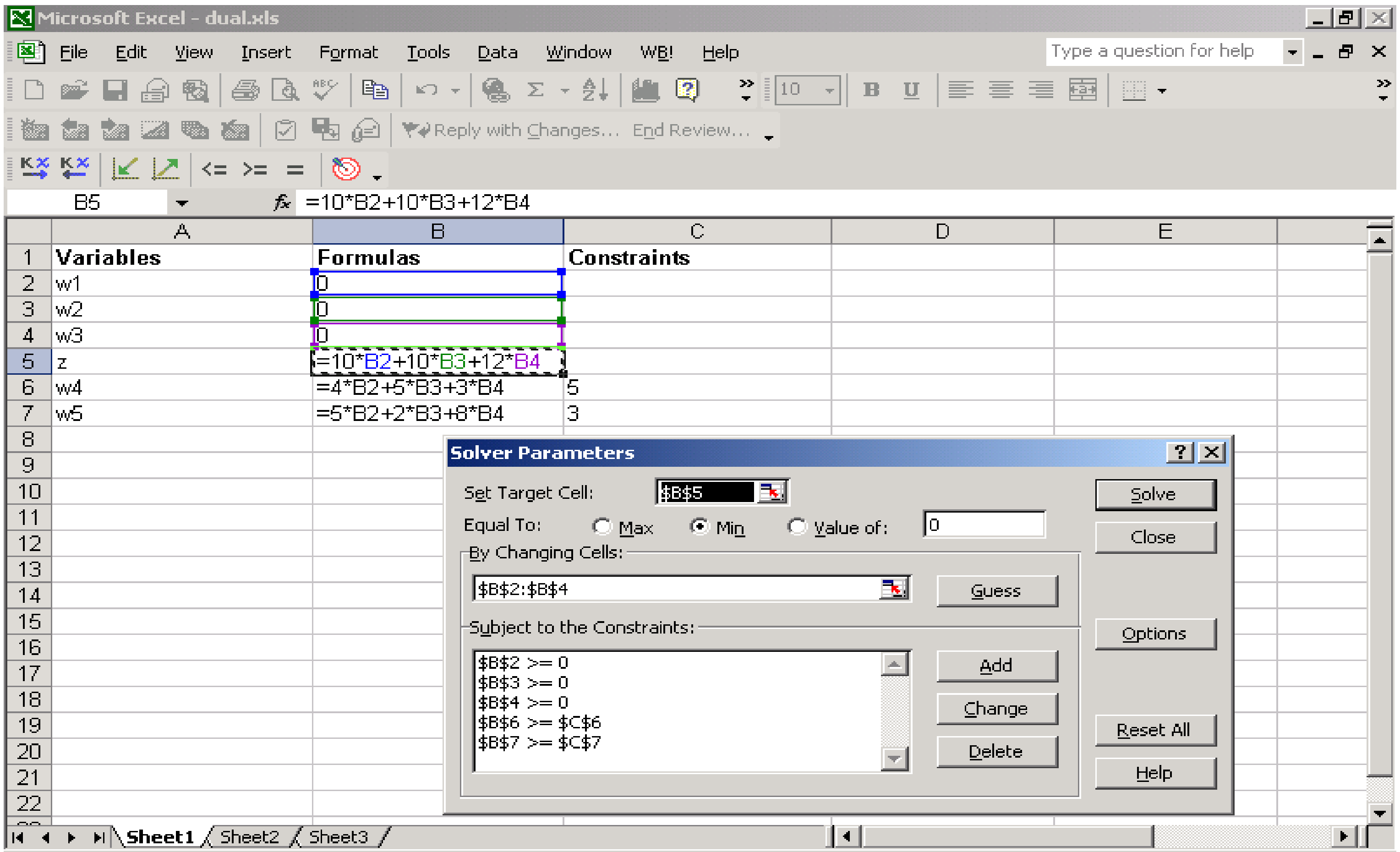
Adjustable Cells

Cell	Name	Original Value	Final Value
$B$2	x1 Formulas	0	1.764705882
$B$3	x2 Formulas	0	0.588235294

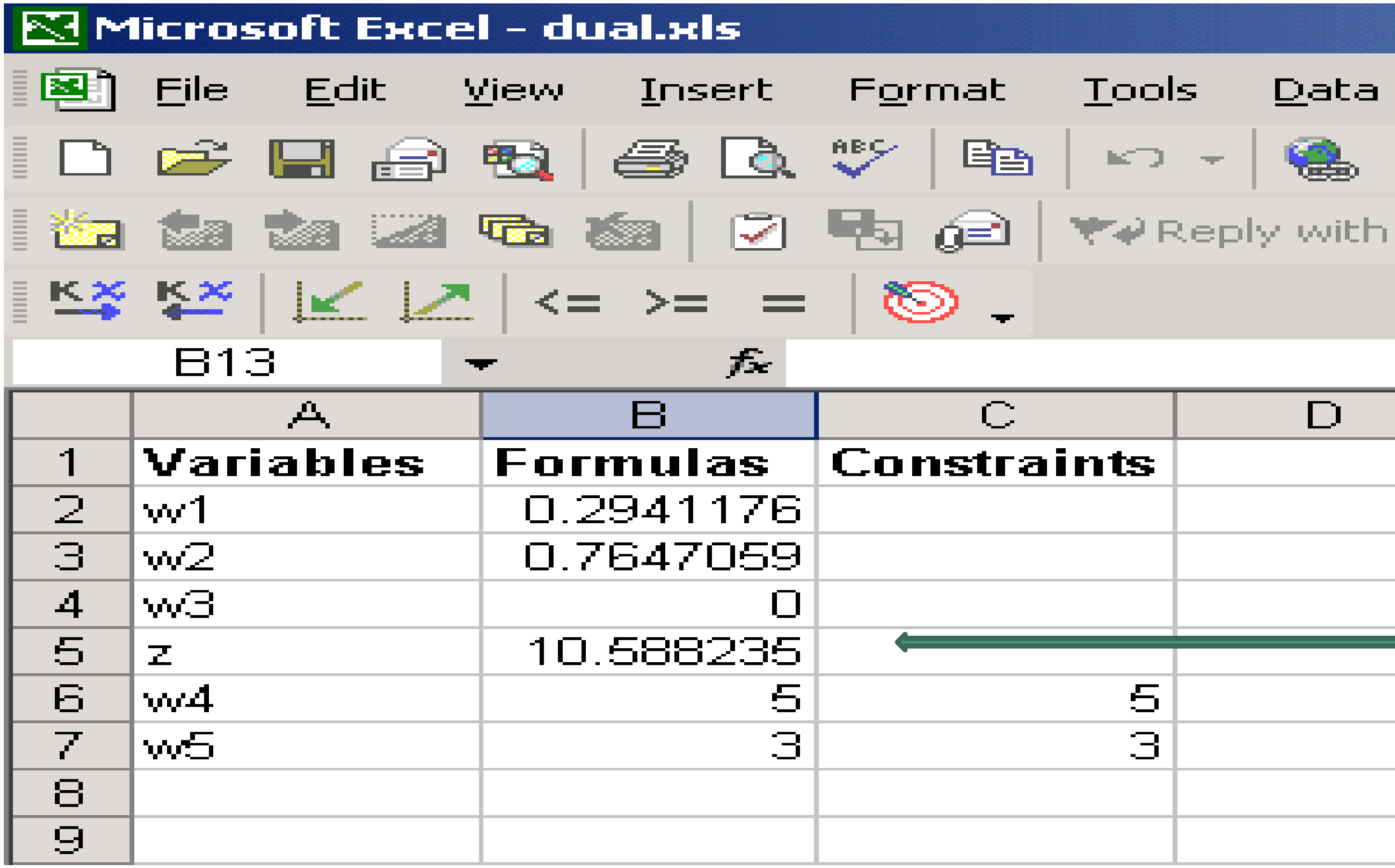
Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$5	x3 Formulas	10	$B$5<=$C$5	Binding	0
$B$6	x4 Formulas	10	$B$6<=$C$6	Binding	0
$B$7	x5 Formulas	10	$B$7<=$C$7	Not Binding	2
$B$2	x1 Formulas	1.764705882	$B$2>=0	Not Binding	1.764705882
$B$3	x2 Formulas	0.588235294	$B$3>=0	Not Binding	0.588235294

Problem in Duality

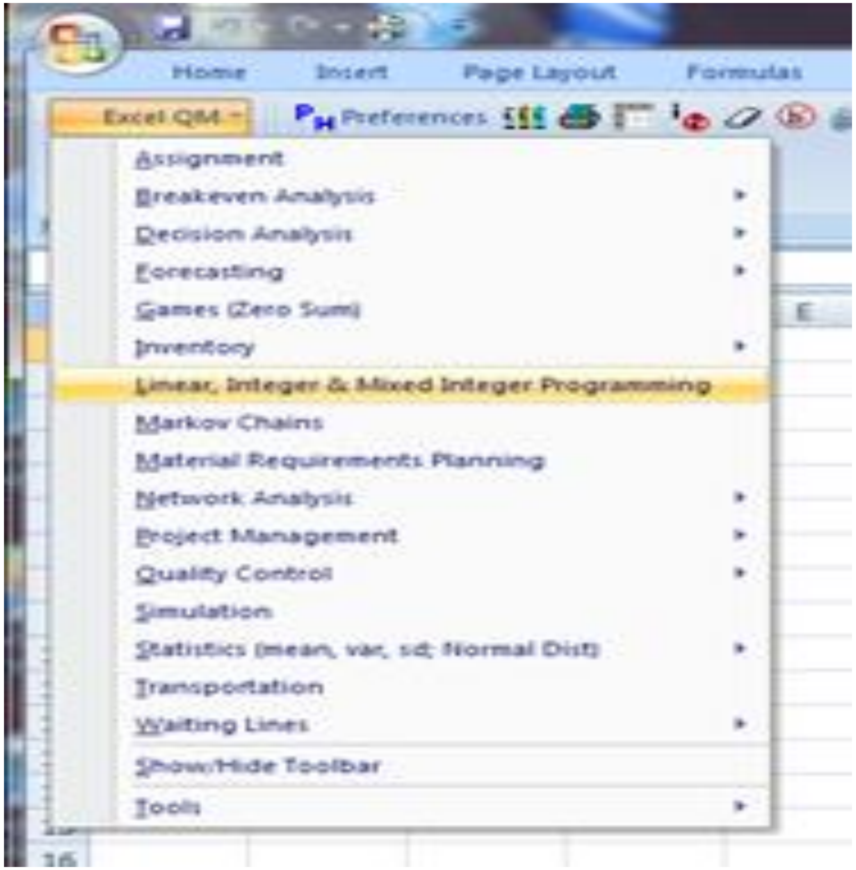


Dual Solution
[z has same value as primal]

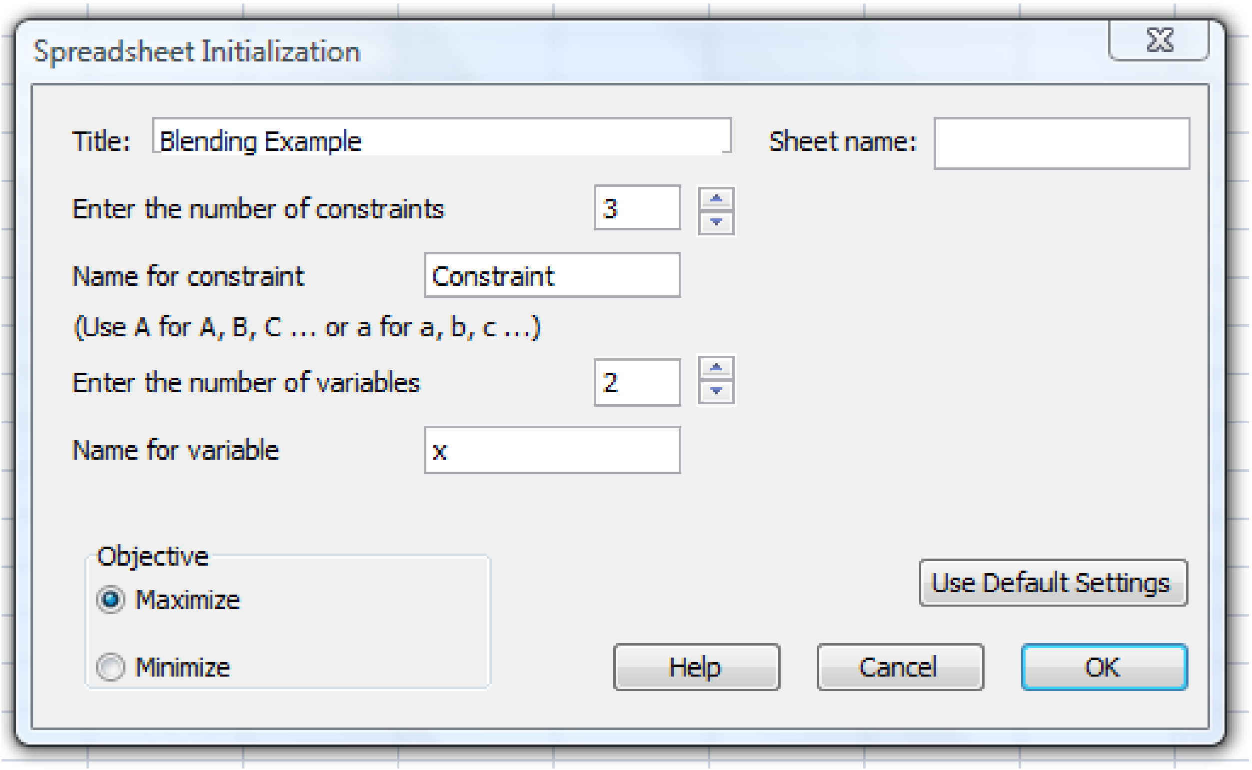


	A	B	C	D
1	Variables	Formulas	Constraints	
2	w1	0.2941176		
3	w2	0.7647059		
4	w3	0		
5	z	10.588235		
6	w4	5	5	
7	w5	3	3	
8				
9				

Using Excel QM

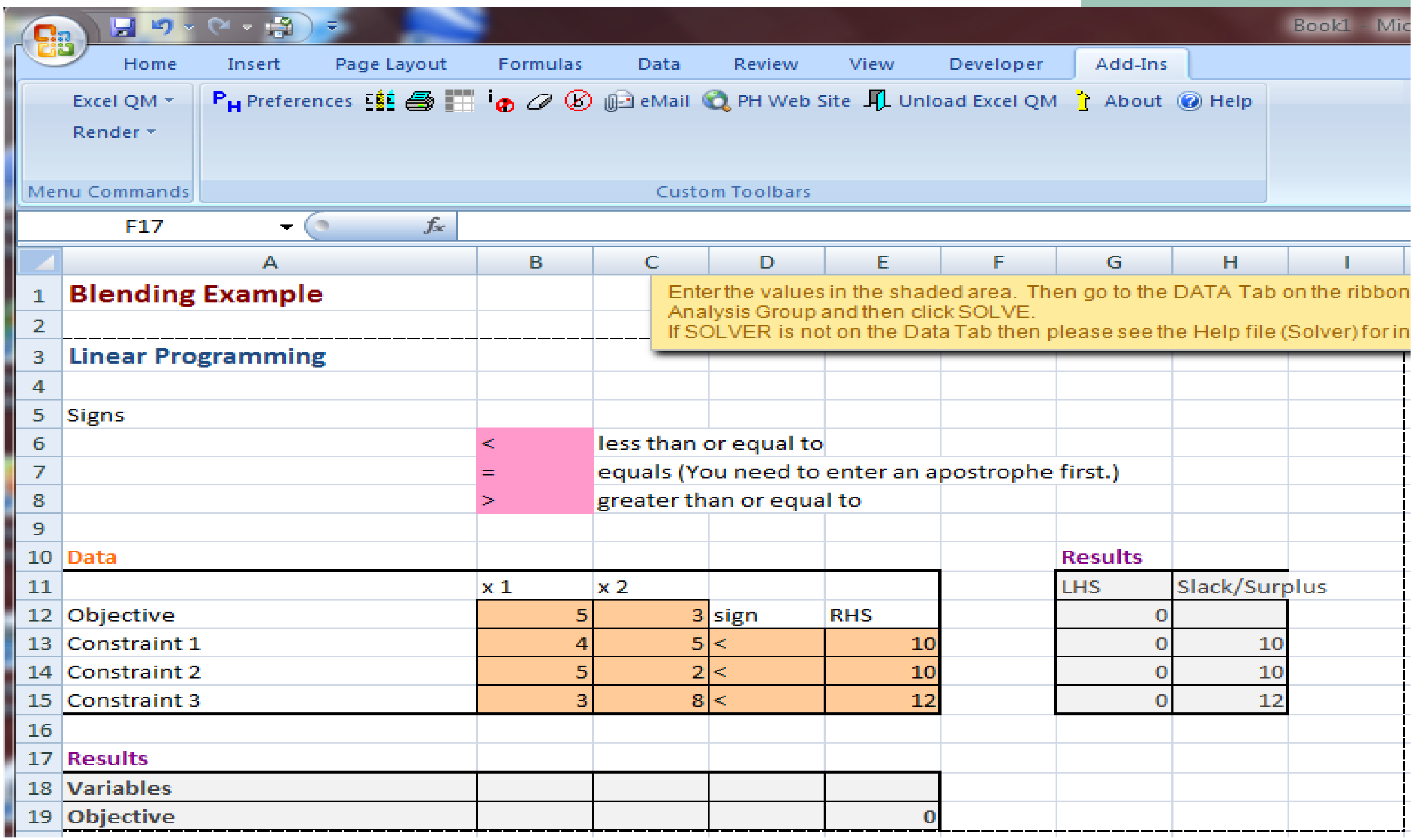


Using Excel QM (con’t)

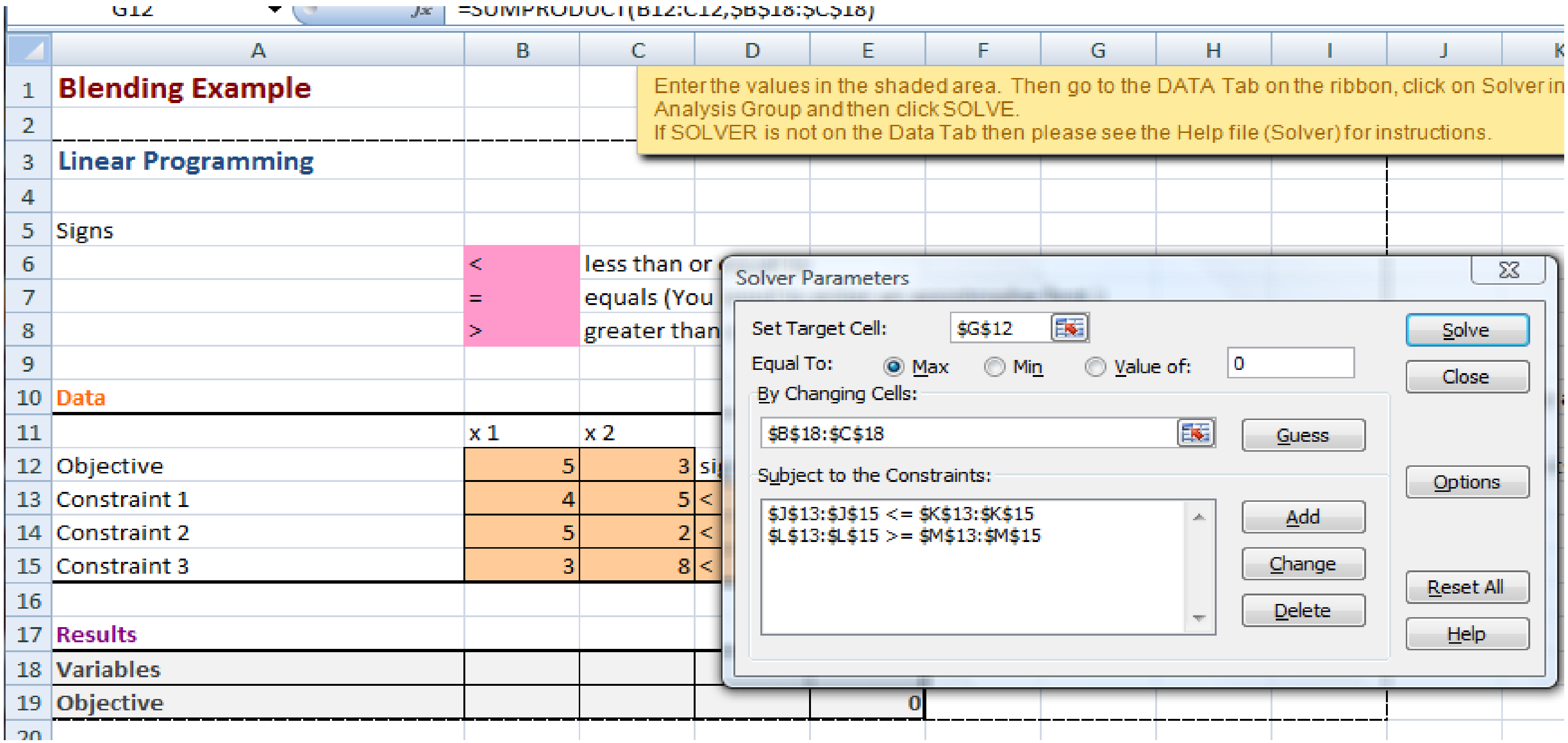


Using Excel QM (con't)

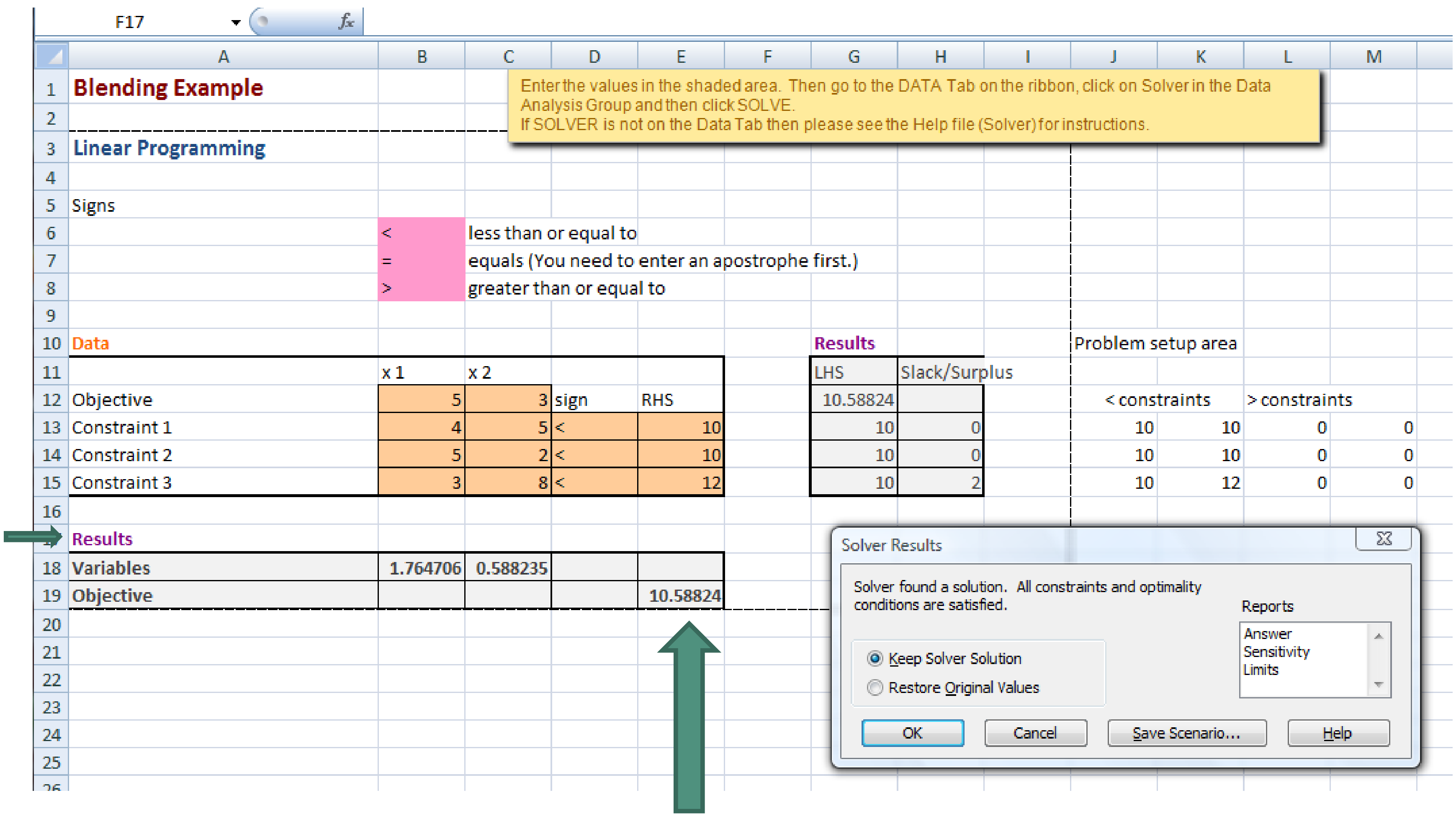
[fill numbers and signs into brown shaded area]



Using Excel QM (con’t)
[after selecting “solver”]



Using Excel QM (con't)



Using QM for Windows On Furniture Problem

- First select the Linear Programming module
- Specify the number of constraints (non-negativity is assumed)
- Specify the number of decision variables
- Specify whether the objective is to be maximized or minimized
- For the Furniture problem there are two constraints, two decision variables, and the objective is to maximize profit

Using QM for Windows

Maximize profit = $\$70T + \$50C$

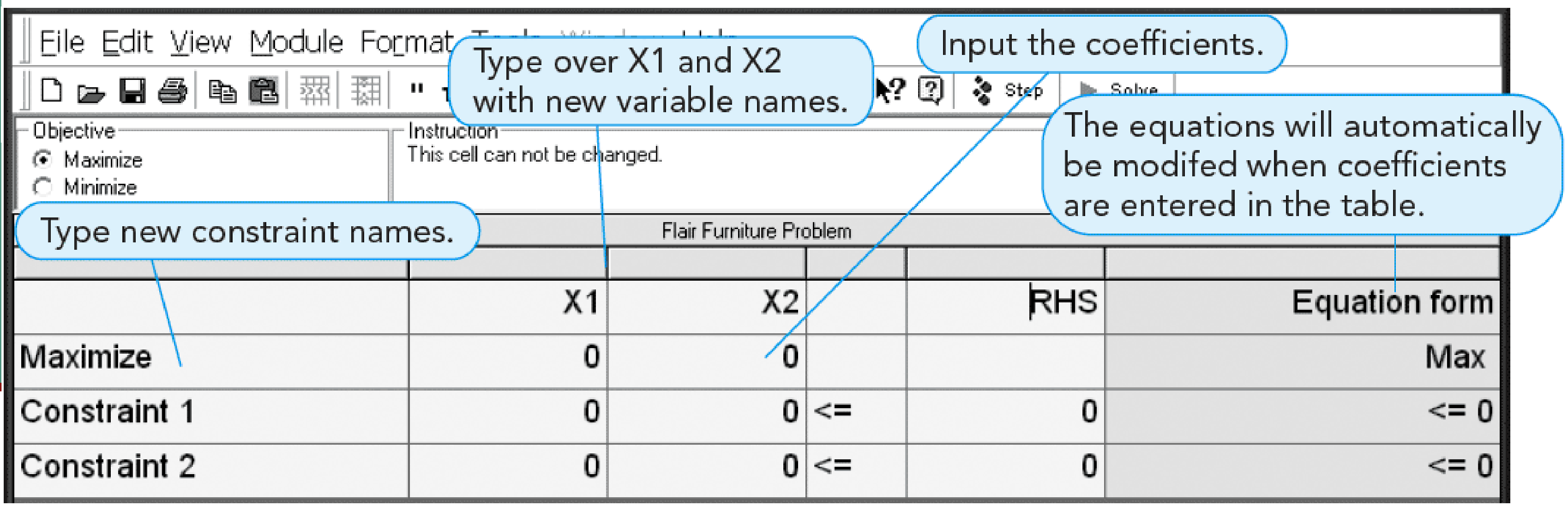
subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)

QM for Windows Linear Programming Computer screen for Input of Data



Using QM for Windows

Maximize profit = $\$70T + \$50C$

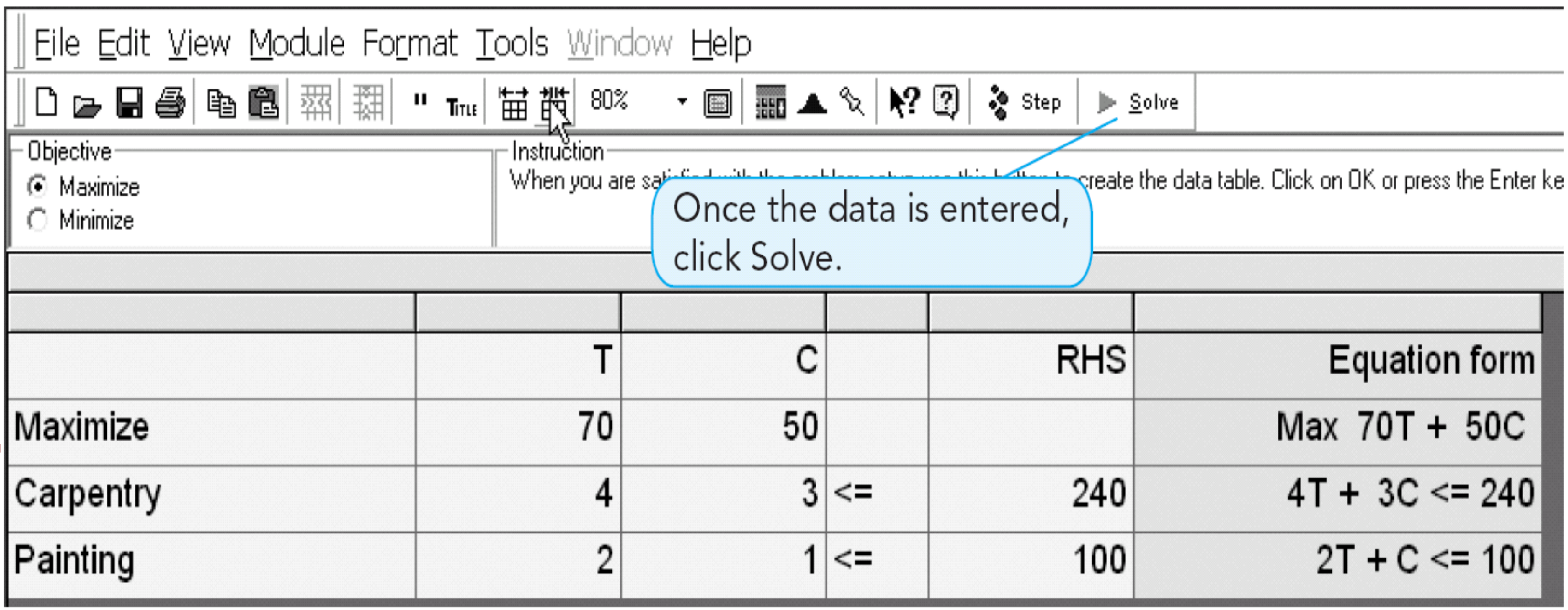
subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)

QM for Windows Data Input for Furniture Problem



	T	C		RHS	Equation form
Maximize	70	50			Max 70T + 50C
Carpentry	4	3	<=	240	4T + 3C <= 240
Painting	2	1	<=	100	2T + C <= 100

Using QM for Windows

Maximize profit = $\$70T + \$50C$

subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)

QM for Windows Output for Furniture Problem

File Edit View Module Format Tools Window Help

Objective: Maximize / Minimize

Instruction: Other output can be viewed by using WINDOW.

Flair Furniture Problem Solution

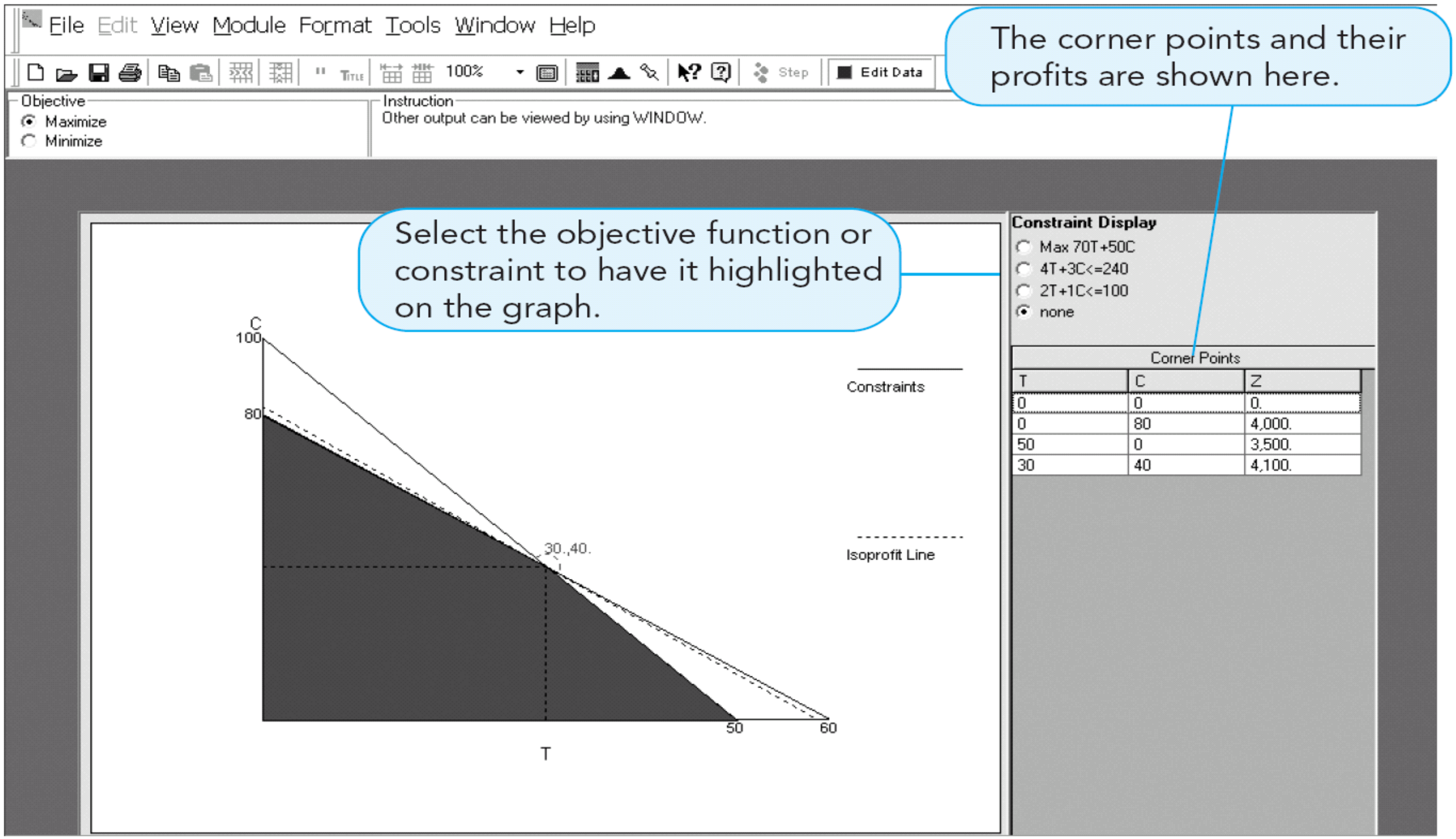
	T	C			Dual
Maximize	70	50			
Carpentry	4	3	<=	240	15
Painting	2	1	<=	100	5
Solution->	30	40		4,100	

The values of the variables are shown here.

The objective function value is shown here.

Using QM for Windows

QM for Windows Graphical Output for Furniture Problem



Using Excel (2010) Solver to Solve the Furniture Problem

- Recall the model for Furniture is:

Maximize profit =	$\$70T + \$50C$		
Subject to	$4T +$	$3C \le 240$	
	$2T +$	$1C \le 100$	

- To use Solver, it is necessary to enter formulas based on the initial model

Using Excel Solver to Solve the Furniture Problem (con't)

1. Enter the variable names, the coefficients for the objective function and constraints, and the right-hand-side values for each of the constraints
2. Designate specific cells for the values of the decision variables
3. Write a formula to calculate the value of the objective function
4. Write a formula to compute the left-hand sides of each of the constraints

Using Excel Solver to Solve the Furniture Problem (con't)

Maximize profit = $\$70T + \$50C$

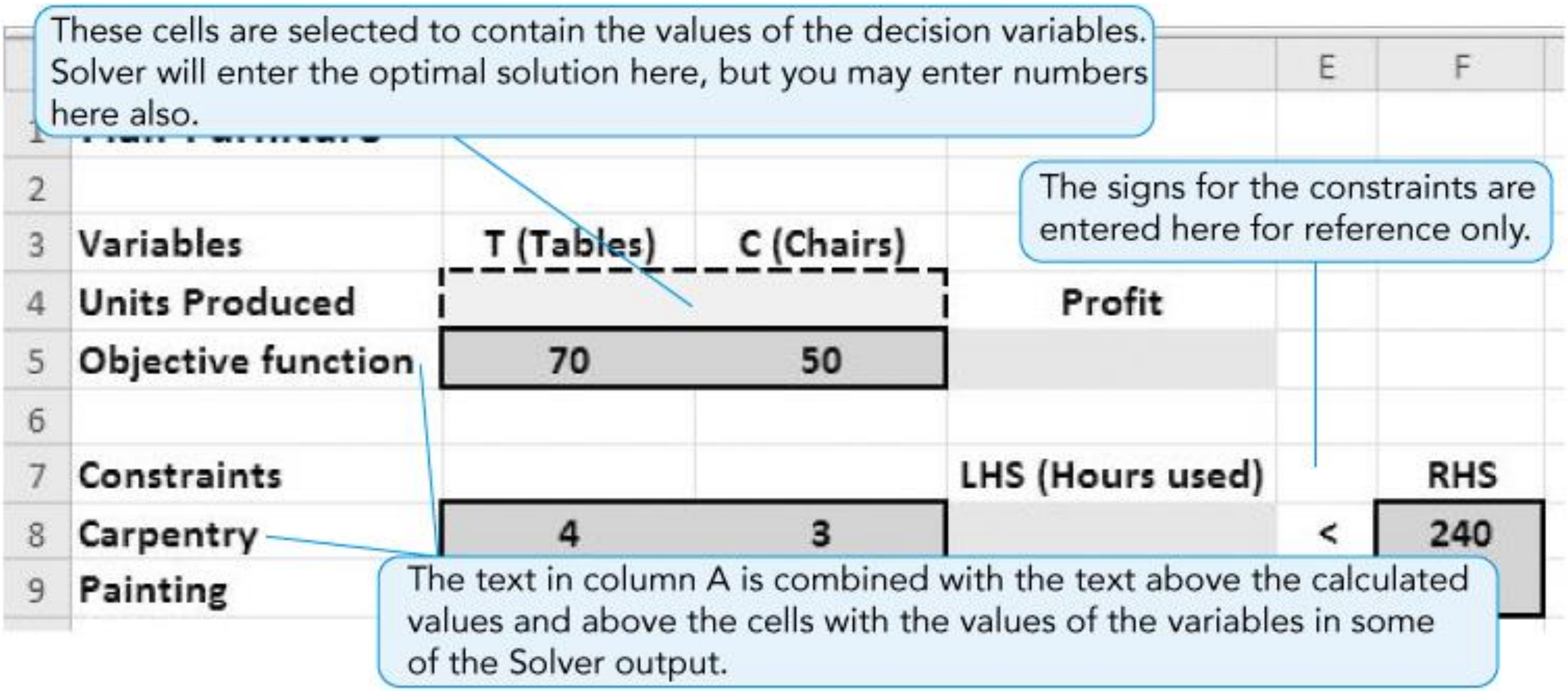
subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)

Excel Data Input for the Furniture Example



Excel Sum of Products Function

	A	B	C	D	E	F	G
1	Employee	Hours	Rate	Total			
2	Elmer	20	25	500	=C2*B2		
3	Leland	40	25	1,000	=C3*B3		
4	Jared	30	25	750	=C4*B4		
5	Damon	35	25	875	=C5*B5		
6	Lloyd	38	25	950	=C6*B6		
7	Hunter	40	25	1,000	=C7*B7		
8	Joshua	21	25	525	=C8*B8		
9				5,600	=SUM(D2:D8)		
10							
11				5,600	=SUMPRODUCT(B2:B8,C2:C8)		
12							

Using Solver to Solve the Furniture Problem (con’t)

Maximize profit = $70T + $50C

subject to

$4T + 3C \le 240$ (carpentry constraint)
$2T + 1C \le 100$ (painting and varnishing constraint)
$T, C \ge 0$ (nonnegativity constraint)

Formulas for the Furniture Example

H16

	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3	**Variables**	**T (Tables)**	**C (Chairs)**			
4	**Units Produced**	**1**	**1**	**Profit**		
5	**Objective function**	**70**	**50**	**=SUMPRODUCT($B$4:$C$4,B5:C5)**		
6						
7	**Constraints**			**LHS (Hours used)**		**RHS**
8	**Carpentry**	**4**	**3**	**=SUMPRODUCT($B$4:$C$4,B8:C8)**	**<**	**240**
9	**Painting**	**2**	**1**	**=SUMPRODUCT($B$4:$C$4,B9:C9)**	**<**	**100**

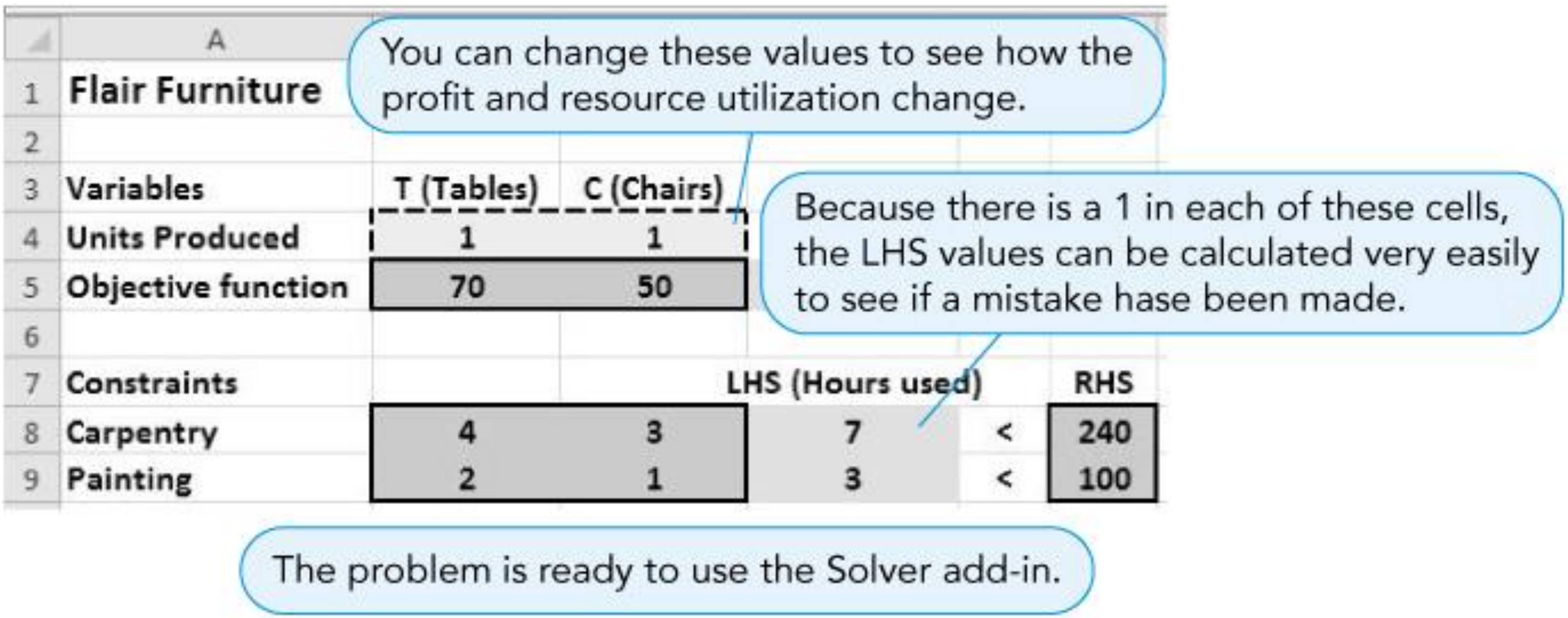
A 1 was entered as the value of T and value of C to help find obvious errors in the formulas.

The values of the variables are in B4 and C4, and the profits for these are in cells B5 and C5. This formula will calculate B4*B5+C4*C5, or 1(70)+1(50), and return a value of 120.

The formula for the LHS of each constraint can be copied from cell D5. The $ signs cause the cell addresses to remain unchanged when the cell (D5) is copied.

Using Solver to Solve the Flair Furniture Problem (con’t)

Excel Spreadsheet for the Flair Furniture Example



	A	B	C	D	E	F
1	Flair Furniture					
2						
3	Variables	T (Tables)	C (Chairs)			
4	Units Produced	1	1			
5	Objective function	70	50			
6						
7	Constraints			LHS (Hours used)		RHS
8	Carpentry	4	3	7	<	240
9	Painting	2	1	3	<	100

Using Solver to Solve the Flair Furniture Problem (con't)

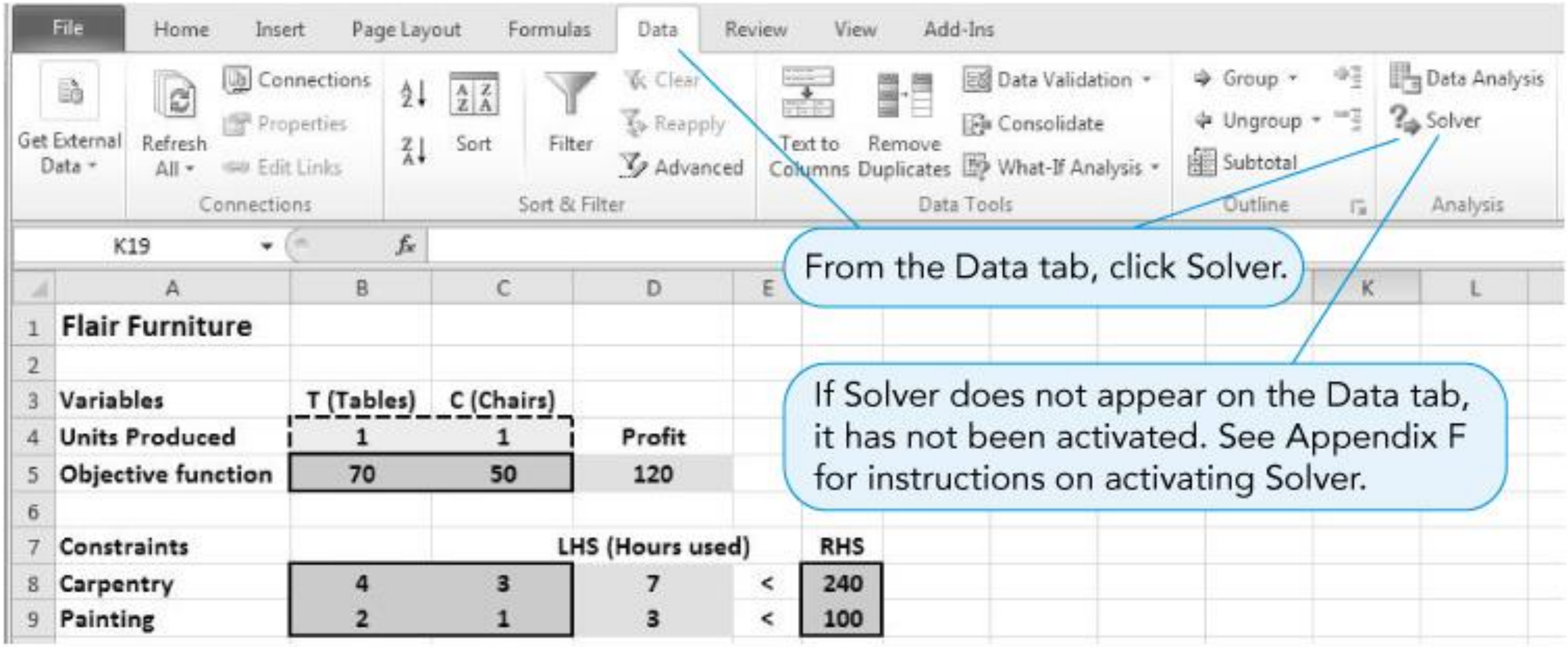
- Once the model has been entered, the following steps can be used to solve the problem

In Excel, select Data –> Solver

1. In the Set Objective box, enter the cell address for the total profit
2. In the By Changing Cells box, enter the cell addresses for the variable values
3. Click *Max* for a maximization problem and *Min* for a minimization problem

Using Solver to Solve the Flair Furniture Problem

Starting Solver



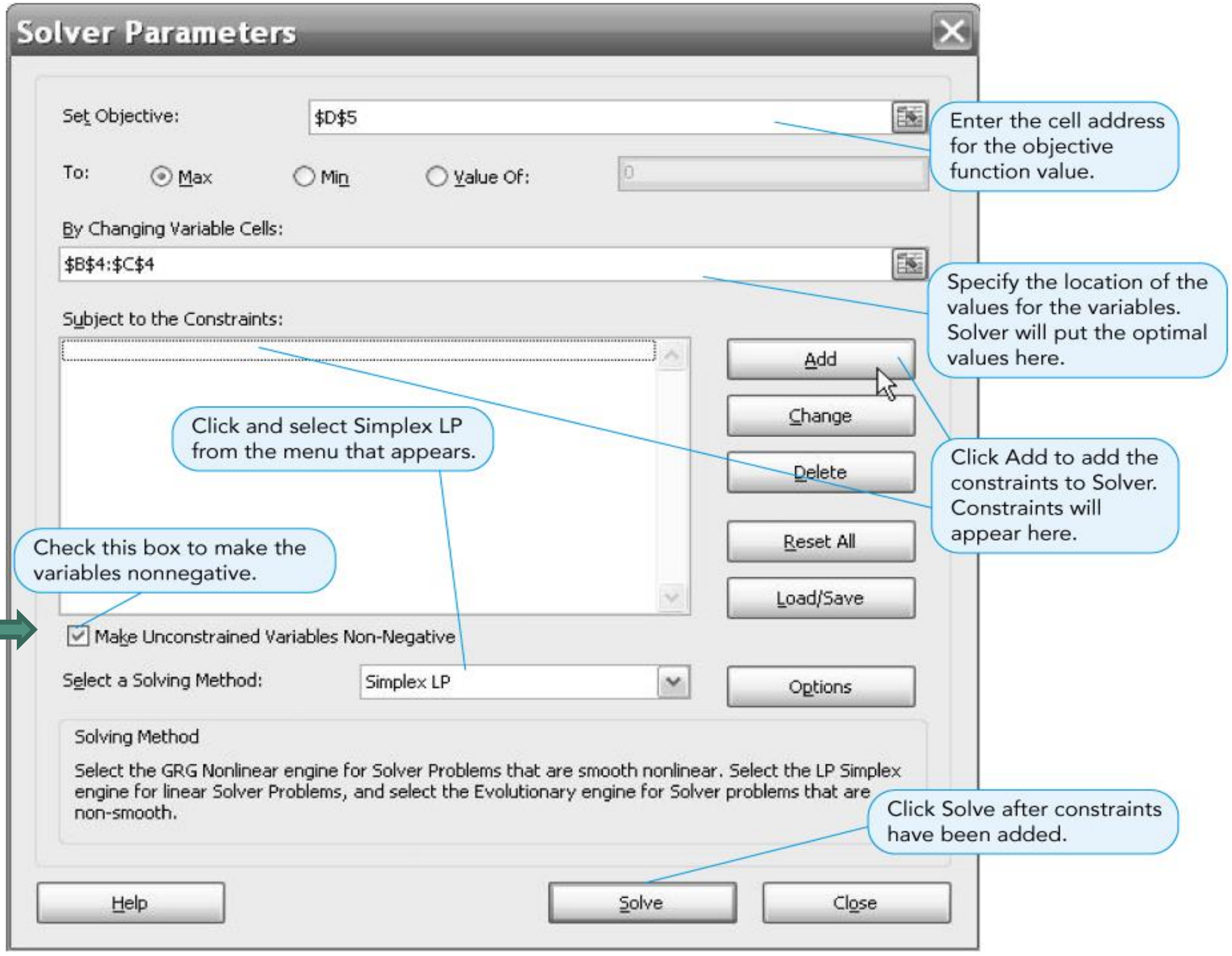
	A	B	C	D	E	F
1	Flair Furniture					
2						
3	Variables	T (Tables)	C (Chairs)			
4	Units Produced	1	1	Profit		
5	Objective function	70	50	120		
6						
7	Constraints			LHS (Hours used)		RHS
8	Carpentry	4	3	7	<	240
9	Painting	2	1	3	<	100

Using Solver to Solve the Flair Furniture Problem (con't)

4. Check the box for *Make Unconstrained Variables Non-negative*
5. Click the *Select Solving Method* button and select *Simplex LP* from the menu that appears
6. Click *Add* to add the constraints
7. In the dialog box that appears, enter the cell references for the left-hand-side values, the type of equation, and the right-hand-side values
8. Click *Solve*

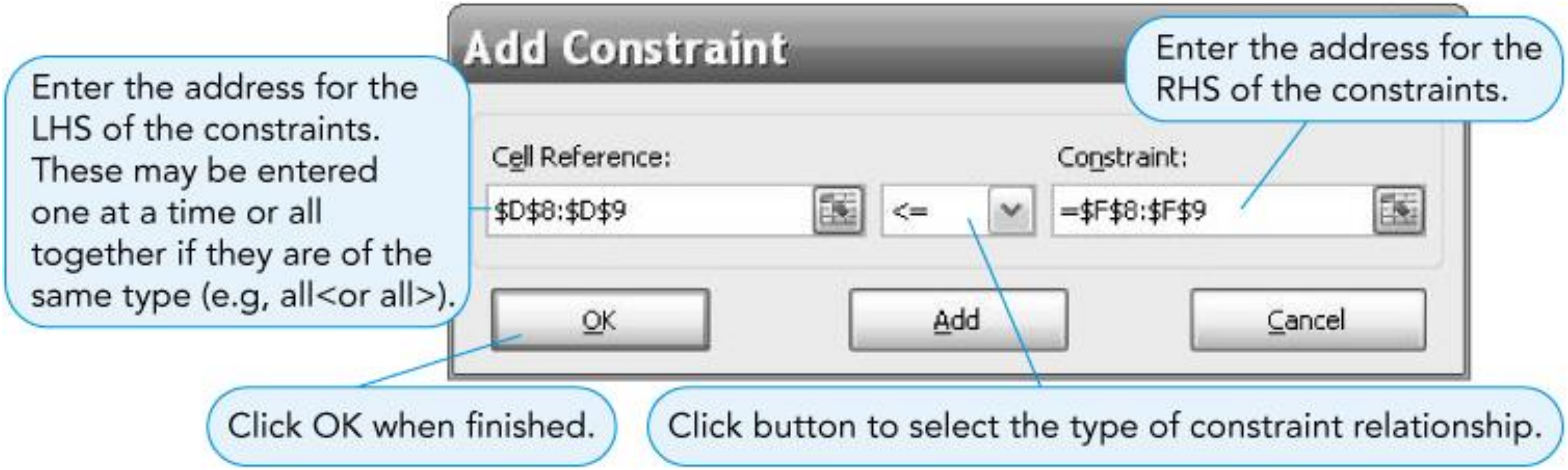
Using Solver to Solve the Flair Furniture Problem (con't)

Solver 2010+ Parameters Dialog Box



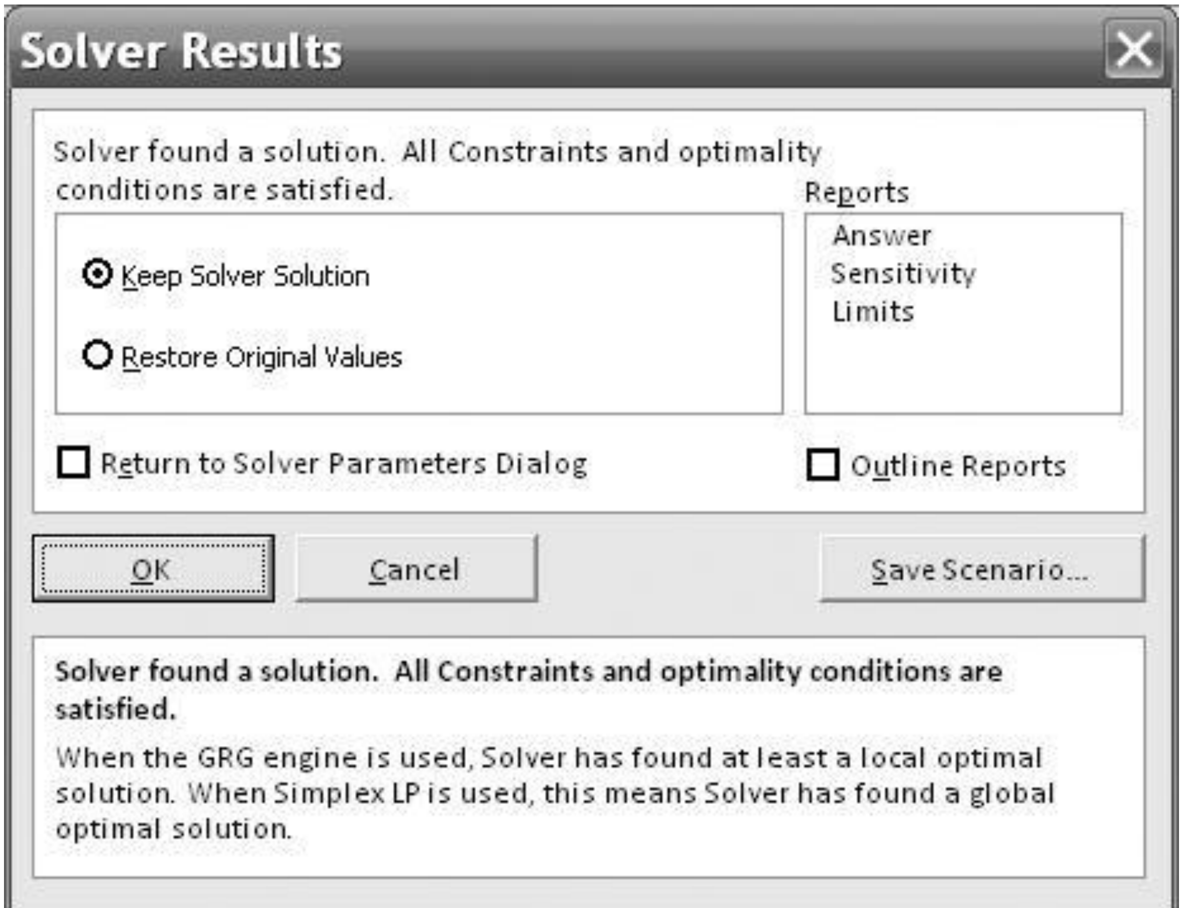
Using Solver to Solve the Flair Furniture Problem (con't)

Solver Add Constraint Dialog Box



Using Solver to Solve the Flair Furniture Problem (con't)

Solver Results Dialog Box



Using Solver to Solve the Flair Furniture Problem (con’t)

Solution Found by Solver

	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3	**Variables**	**T (Tables)**	**C (Chairs)**			
4	**Units Produced**	**30**	**40**	**Profit**		
5	**Objective function**	**70**	**50**	**4100**		
6						
7	**Constraints**			**LHS (Hours used)**		**RHS**
8	**Carpentry**	**4**	**3**	**240**	**<**	**240**
9	**Painting**	**2**	**1**	**100**	**<**	**100**

The optimal solution is *T*=30, *C*=40, profit=4100.

The hours used are given here.

Equality Constraints

- Equality constraints can also be handled via LP methods
- Suppose there were a constraint that the ratio of the number of chairs to tables produced had to be 4
- The constraint would be: $C/T = 4$
- This would be transformed into:
  - $C = 4T$
  - $C - 4T = 0$

Sensitivity Analysis

- We have assumed complete certainty in the data and relationships of a problem
- But in the real world, conditions are dynamic and changing
- We can analyze how *sensitive* a deterministic solution is to changes in the assumptions of the model
- This is called *sensitivity analysis*, *postoptimality analysis*, *parametric programming*, or *optimality analysis*

Sensitivity Analysis (con't)

- Sensitivity analysis often involves a series of what-if questions concerning constraints, variable coefficients, and the objective function
- One way to do this is the trial-and-error method where values are changed and the entire model is resolved
- The preferred way is to use an analytic postoptimality analysis
- After a problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution

Changes in the Objective Function Coefficient

- In real-life problems, contribution rates (cost coefficients) in the objective functions fluctuate periodically
- Graphically, this means that although the feasible solution region remains exactly the same, the slope of the isoprofit line will change
- We can often make modest increases or decreases in the objective function coefficient of any variable without changing the current optimal corner point
- The allowable increase/decrease shows how much change can be made before the optimal point is on another vertex
- The reduced cost (or reduced gradient) shows how much the objective function will decrease if a zero variable is increased

Changes in the Objective Function Coefficient (con't)

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Book2]problem
Report Created: 4/6/2010 2:48:23 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$18	Variables x 1	1.764705882	0	5	2.5	2.6
$C$18	Variables x 2	0.588235294	0	3	3.25	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$J$13	<<constraints	10	0.294117647	10	2.473684211	2
$J$14	<<constraints	10	0.764705882	10	2.5	3.481481481
$J$15	<<constraints	6.470588235	0	12	1E+30	5.529411765
$L$13	<>constraints	0	0	0	0	1E+30
$L$14	<>constraints	0	0	0	0	1E+30
$L$15	<>constraints	0	0	0	0	1E+30

Changes in Resources or Right-Hand-Side Values

- The right-hand-side values of the constraints often represent resources available to the firm
- If additional resources were available, a higher total profit could be realized
- Sensitivity analysis about resources will help answer questions about how much should be paid for additional resources and how much more of a resource would be useful

Changes in Resources or Right-Hand-Side Values (con't)

- If the right-hand side of a constraint is changed, the feasible region will change (unless the constraint is redundant)
- Often the optimal solution will change
- The amount of change in the objective function value that results from a unit change in one of the resources available is called the shadow price (or *dual price* or *lagrange multiplier)*
- The shadow price for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint

Shadow Prices

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Book2]problem
Report Created: 4/6/2010 2:48:23 PM

Adjustable Cells

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$L$13	<> constraints	0	0	0	0	1E+30
$L$14	<> constraints	0	0	0	0	1E+30
$L$15	<> constraints	0	0	0	0	1E+30

Changes in Resources or Right-Hand-Side Values (con't)

- However, the amount of possible increase in the right-hand side of a resource is limited
- If the number of hours increased beyond the upper bound, then the objective function would no longer increase by the shadow price
- There would simply be excess (*slack*) hours of a resource or the objective function may change by an amount different from the shadow price
- The shadow price is relevant only within limits

Shadow Price Limits (prior to slack)

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Book2]problem
Report Created: 4/6/2010 2:48:23 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$18	Variables x 1	1.764705882	0	5	2.5	2.6
$C$18	Variables x 2	0.588235294	0	3	3.25	1

Constraints

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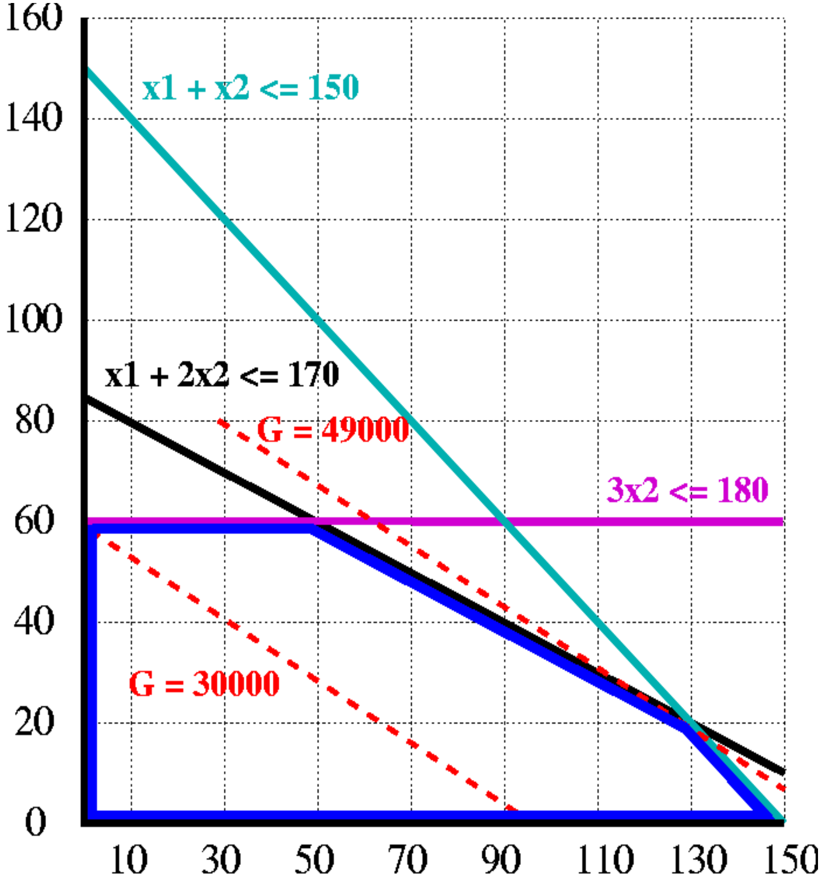
$L$13	<>constraints	0	0	0	0	1E+30
$L$14	<>constraints	0	0	0	0	1E+30
$L$15	<>constraints	0	0	0	0	1E+30

Changes in the Technological Coefficients

- Changes in the technological (constraint) coefficients often reflect changes in the state of technology
- If the amount of resources needed to produce a product changes, coefficients in the constraint equations will change
- This does not change the objective function, but it can produce a significant change in the shape of the feasible region
- This may also cause a change in the optimal solution

Problem Formulation

- Today the main difficulty is <u>formulating</u> the LP problem, not in solving it !!!

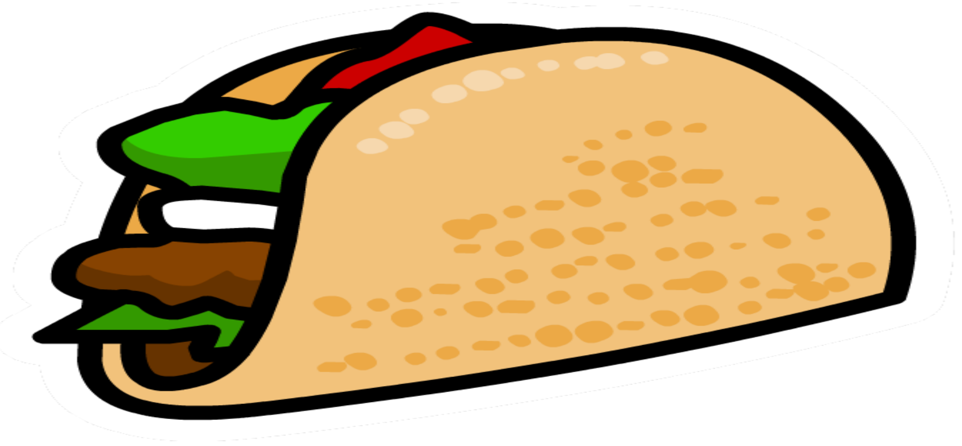


Diet or Feed Mix Problem

- Finding the optimal (typically lest costly) mix of ingredients
- That satisfy nutritional and other requirements

Diet or Feed Mix Problem (con't)

- Taco Mania wants to find the lowest cost set of ingredients for its new "Rancho Whatever" dish
- The dish is made from chicken, rice, and beans (not including shell)



Diet or Feed Mix Problem (con't)

- The composition of the ingredients is:
  - Chicken
    - $ 1 per pound
    - 80% protein
    - 20% fat
  - Rice
    - $.30 per pound
    - 30% protein
    - 70% carbs
  - Beans
    - $.50 per pound
    - 40% protein
    - 30% carbs
    - 30% fat

Diet or Feed Mix Problem (con't)

- Find the mix that yields the <u>lowest cost</u> subject to:
  - 1 pound serving
  - Protein: greater that 40%
  - Fat: less than 20%, greater than 10%
  - Carbs: less than 70%, greater than 40%

- What are the variables ?
- What is the objective function?
- What are the constraints?

Thought ???

- Do not look ahead !

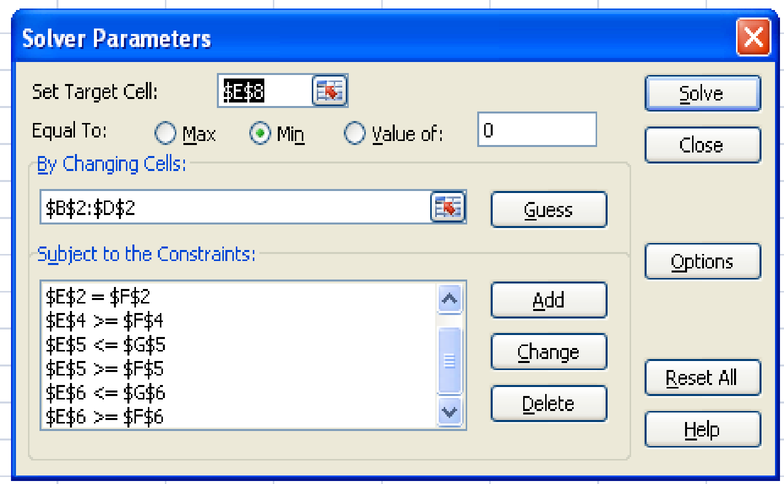
Diet or Feed Mix Problem (con't)

- Minimize Cost = C + .3*R + .5*B
  - C=lb of chicken, R=lb of rice, B=lb of beans
- Constraints:
  - One pound: C + R + B = 1
  - Protein: .8*C + .3*R + .4*B >= .4
  - Carb: .7*R + .3*B >= .4
  - .7*R + .3*B <= .7
  - Fat : .2*C + .3*B >= .1
  - .2*C + .3*B <= .2

Diet or Feed Mix Problem (con't)

E8 =B8*B2+C8*C2+D8*D2

	A	B	C	D	E	F	G
1		Chicken	Rice	Beans			
2	Serving	0.153846	0.615385	0.230769	1	1	
3							
4	Protein	0.8	0.3	0.4	0.4	0.4	
5	Carb	0	0.7	0.3	0.5	0.4	0.7
6	Fat	0.2	0	0.3	0.1	0.1	0.2
7						Min	Max
8	Cost	1	0.3	0.5	0.453846		

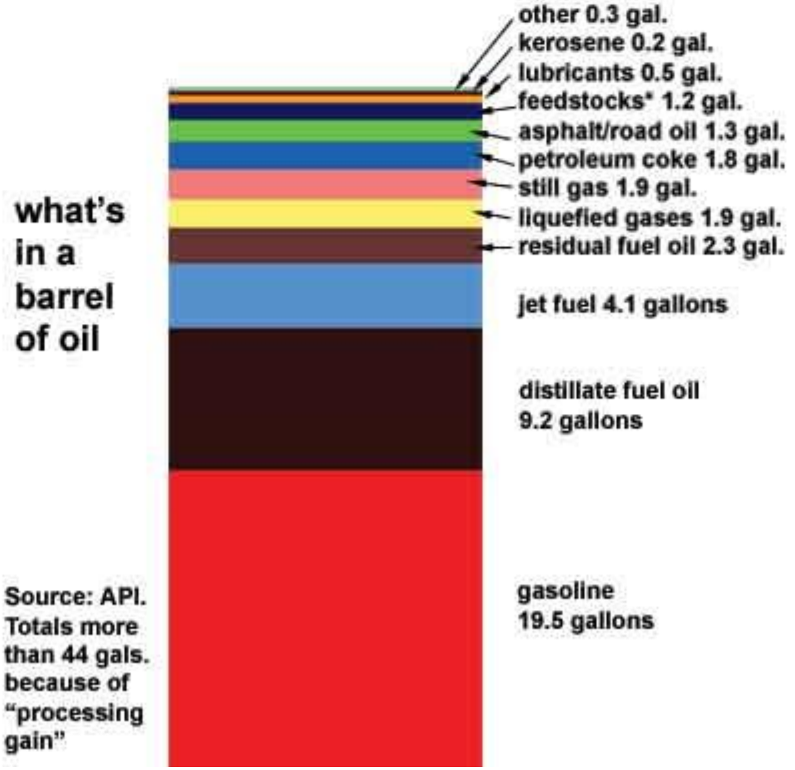


Diet or Feed Mix Problem (con't)

	A	B	C	D	E	F	G
1		Chicken	Rice	Beans			
2	Serving	0.15384614979663	0.615384616734457	0.230769233468913	=B2+C2+D2	1	
3							
4	Protein	0.8	0.3	0.4	=B4*B2+C4*C2+D4*D2	0.4	
5	Carb	0	0.7	0.3	=B5*B2+C5*C2+D5*D2	0.4	0.7
6	Fat	0.2	0	0.3	=B6*B2+C6*C2+D6*D2	0.1	0.2
7						Min	Max
8	Cost	1	0.3	0.5	=B8*B2+C8*C2+D8*D2		
9							
10							
11		B2 is pounds of chicken					
12		C2 is pounds of rice					
13		D2 is pounds of beans					
14							

Blending Problem

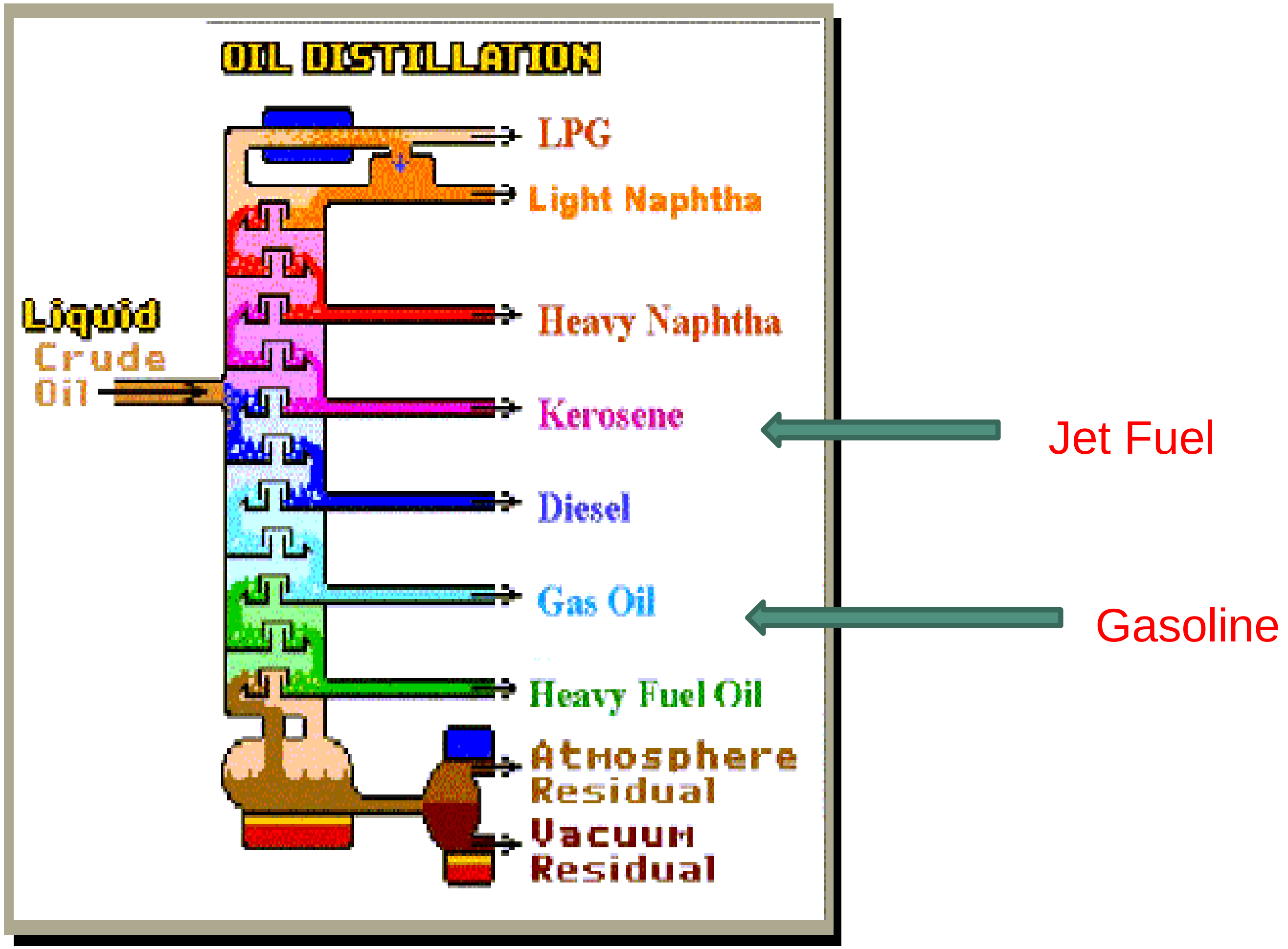
- Blending two or more raw materials to make one or more products
- That satisfy raw material availability constraints and product composition constraints



Blending Problem (con't)

- Fuel production (fuel is highest cost for airlines)
  - Transport crude to refineries (boat, pipeline)
    - Cheaper to transport and store crude rather than transport finished products (gasoline, jet fuel, etc.)
  - Inventory crude in container farms – optimize inventory (holding cost vs cost of shortage)
  - Distillation – crude distillation tower
  - Catalytic cracking
  - Inventory finished product (to supply demand)
  - Transport finish product to market (pipeline, truck, rail)

Crude Oil Distillation



Blending Problem (con't)

Refinery in Wood River, Illinois. Crude travels by barge up Mississippi River from Gulf of Mexico. Built in 1917 by Shell Oil, bought by Conoco Philips in about 2004.

The facility produces 165,000 barrels per day of gasoline, 90,000 barrels per day of diesel and jet fuels, as well as other products.

Blending Problem (con't)

- Both regular and economy gasoline is made from a mixture of crude 1 and crude 2 oils
- Both crude 1 and crude 2 contain ingredients A and B (and other hydrocarbons):
  - Crude 1 is 35% A and 55% B and cost $30/barrel
  - Crude 2 is 60% A and 25% B and cost $34.80/barrel
- Regular must have at least 45% ingredient A, and economy be at most 50% of ingredient B
- Demand for regular is 25,000 and for economy is 32,000; demand must be satisfied
- **What are the variables, the objective function, and the constraints ?**

Thought ???

- Do not look ahead !

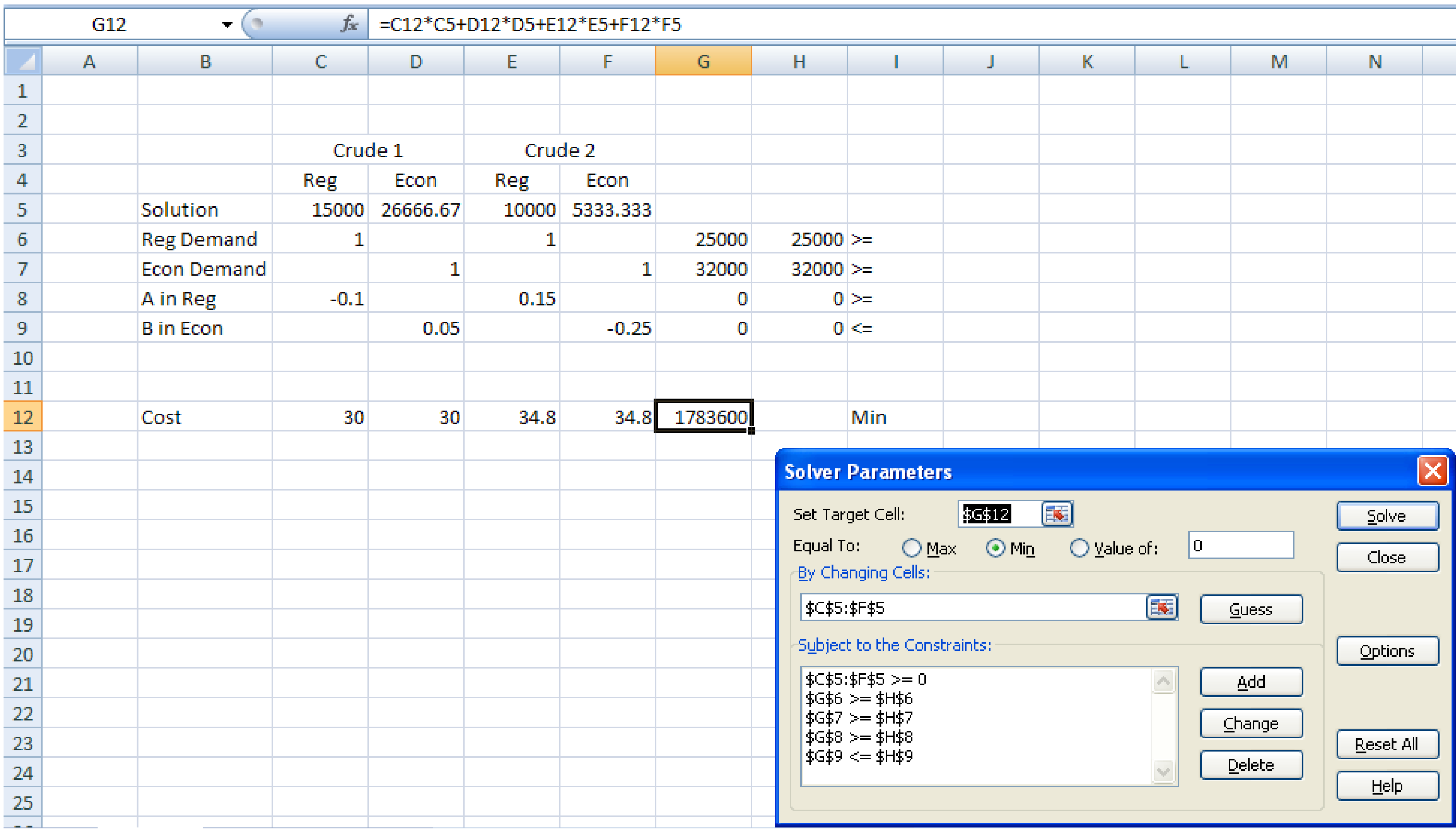
Blending Problem (con't)

- Variables:
  - Regular gas: crude 1 (R1), crude 2 (R2)
  - Economy gas: crude 1 (E1), crude 2 (E2)
- Cost: 30*R1 + 34.8*R2 + 30*E1 + 34.8*E2
- Constraints:
  - Demand: Regular: R1 + R2 >= 25000
  - Economy: E1 + E2 >= 32000
  - Ingredient A in regular >= .45
  - Ingredient B in economy <= .5

Blending Problem (con’t)

- Crude 1 is 35% A and 55% B
- Crude 2 is 60% A and 25% B
- 45% A in regular constraint
  - Available A is .35 * of Crude1 + .60 * of Crude 2
  - Used <= Available
  - .45 * (R1 + R2) <= .35 * R1 + .60 * R2
  - -.1 * R1 + .15 * R2 >= 0
- 50% B in economy constraint
  - Available B is .55 * of Crude1 + .25 * of Crude2
  - .5 * (E1 + E2) <= .55 * E1 + .25 * of E2
  - .5 * E1 - .25 * E2 <= 0

Blending Problems (con’t)



Blending Problems (con't)

J21

	A	B	C	D	E	F	G	H	I
1									
2									
3			Crude 1		Crude 2				
4			Reg	Econ	Reg	Econ			
5		Solution	15000	26666.6666666667	10000	5333.33333333333			
6		Reg Demand	1		1		=C6*C5+E6*E5	25000	>=
7		Econ Demand		1		1	=D7*D5+F7*F5	32000	>=
8		A in Reg	-0.1		0.15		=C8*C5+E8*E5	0	>=
9		B in Econ		0.05		-0.25	=D9*D5+F9*F5	0	<=
10									
11									
12		Cost	30	30	34.8	34.8	=C12*C5+D12*D5+E1		Min
13									
14									

Production Planning Problem

- How many units should be produced each month during the "planning horizon" (typically one year, but in our example here we'll only do four months)
- Marketing will forecast a demand by month
- We want to minimize the total cost which is the sum of production cost plus inventory cost
- Production costs vary considerably by month since labor and fuel cost are seasonal
- Production capacity will typically also vary by month since alternative products use some of the same production resources

Planning Data

Month	Demand (units)	Cost ($/unit)	Capacity (units)
Jan	30	2.10	90
Feb	50	3.20	90
Mar	150	1.50	120
Apr	50	4.75	100

To satisfy demand we could produce 30 in Jan, 50 in Feb, 150 in Mar, etc.; that way there would be no inventory carrying cost. But what is wrong with that approach ?

Month	Demand (units)	Cost ($/unit)	Capacity (units)
Jan	30	2.10	90
Feb	50	3.20	90
Mar	150	1.50	120
Apr	50	4.75	100

Production Planning (con't)

- We have capacity constrains, and production costs may be lower at other times
- Initial condition:
  - 10 units are in stock at the beginning of January
- Ending condition:
  - At least 15 units should be in stock at the end of April
- Inventory holding cost:
  - $ 0.20 per unit per month

Production Planning (con't)

- How many units should be produced each month:
  - So that demand is satisfied
  - Cost is minimized
- Cost is the sum of:
  - Production cost
  - Inventory holding cost
- What are the variables ?

Variables

- X_i = production in month i
  - X1 = Jan
  - X2 = Feb
  - X3 = Mar
  - X4 = Apr

- I_i = inventory at <u>end</u> month i
  - I1 = Jan
  - I2 = Feb
  - I3 = Mar
  - I4 = Apr

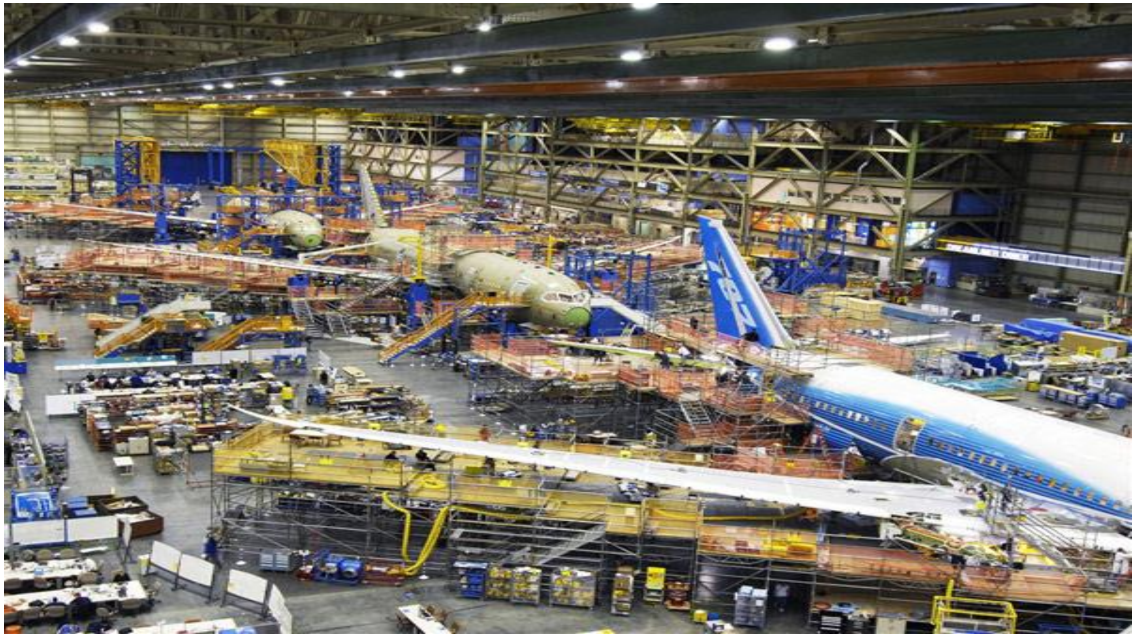
- **What is the formula for the inventory at the end of each month in terms of the production and demand for that month ?**

- Do not look ahead !

Inventory Formulas

- Inventory at the end of a month is the beginning inventory (end of prior month) plus the production for that month minus the sales (demand):
  - $I_i = I_{i-1} + X_i - S_i$
- Thus ending inventory:
  - $I_N = I_0 + \sum (X_i - S_i)$ where the sum is from i=1 to N

- What is the objective function (to be minimized) ?



- Do not look ahead !

Objective Function

- Minimize:
  - Sum of production cost plus
    - Sum of monthly production times the production cost that month
  - Sum of inventory cost
    - Inventory level times holding cost per unit

- $Z = 2.1X1 + 3.2X2 + 1.5X3 + 4.75X4 + 0.2*(I1 + I2 + I3 + I4)$

- What are the constraints ?

Constraints

- Capacity:
  - Xi <= Ci
- Equations for ending Inventory:
  - $I_i = I_{i-1} + X_i – S_i$
- Ending condition:
  - $I_4 >= 15$
- Demand satisfied:
  - $I_{i-1} + X_i >= S_i$ or simply $I_i >= 0$

Excel Solution

	A	B	C	D	E	F	G	H
2			Beginning	Units	Ending			Holding Cost/
3	Month	Demand	Inventory	Produced	Inventory	Capacity	Cost/Unit	Unit/Month
4	Jan	30	10	165	145	166	$2.10	$0.20
5	Feb	50	145	0	95	90	$3.20	$0.20
6	Mar	150	95	120	65	120	$1.50	$0.20
7	Apr	50	65	0	15	100	$4.75	$0.20
8	Totals	280				476		
10	Minumum Final Inventory:			15			Total Cost:	$590.50

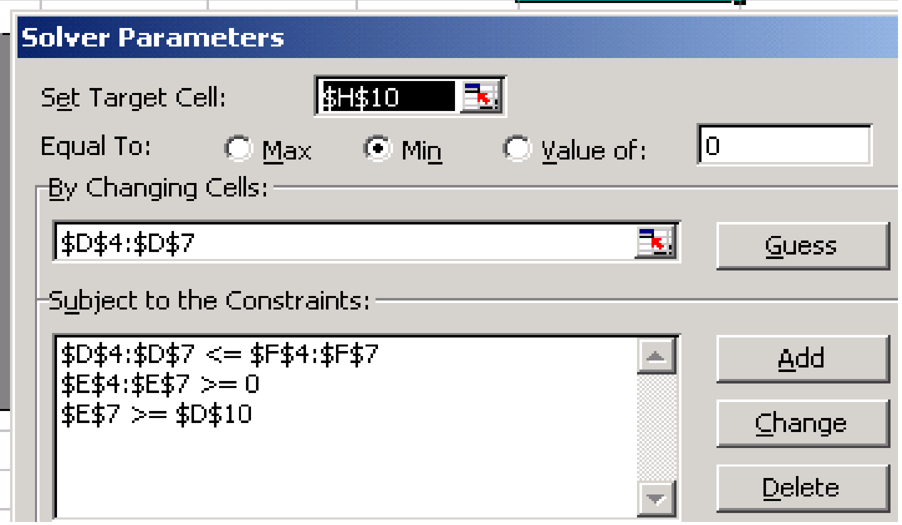
Solver Model:

Target Cell: $H$11

Changing Cells: $D$5:$D$8

Constraints:

- $D$5:$D$8 <= $F$5:$F$8
- $E$5:$E$8 >= 0
- $E$8 >= $D$11
- All variables non-negative



Portfolio Applications

- Bank, investment funds, and insurance companies often have to select specific investments from a variety of alternatives
- The manager's overall objective is generally to maximize the potential return on the investment given a set of legal, policy, or risk restraints

Portfolio Applications (con't)

- International City Trust (ICT) invests in short-term trade credits, corporate bonds, gold stocks, and construction loans
- The board of directors has placed limits on how much can be invested in each area:

INVESTMENT	INTEREST EARNED (%)	MAXIMUM INVESTMENT ($ MILLIONS)
Trade credit	7	1.0
Corporate bonds	11	2.5
Gold stocks	19	1.5
Construction loans	15	1.8

Portfolio Applications (con't)

- ICT has $5 million to invest and wants to accomplish two things:
  - Maximize the return on investment over the next six months
  - Satisfy the diversification requirements set by the board
- The board has also decided that at least 55% of the funds must be invested in gold stocks and construction loans and no less than 15% be invested in trade credit
- **What are the variables ?**

Portfolio Applications (con't)

The variables in the model are:

X_1 = dollars invested in trade credit
X_2 = dollars invested in corporate bonds
X_3 = dollars invested in gold stocks
X_4 = dollars invested in construction loans

What are the constraints ?

Portfolio Applications (con't)

- $ 5 million total invested
- Limits on amount invested in each category
- 55% of the funds must be invested in gold stocks and construction loans
- No less than 15% be invested in trade credit
- What is the objective function ?

Portfolio Applications (con't)

Objective:

Maximize dollars of interest earned $= 0.07X_1 + 0.11X_2 + 0.19X_3 + 0.15X_4$

subject to:

$$
\begin{aligned}
X_1 &\le 1{,}000{,}000 \\
X_2 &\le 2{,}500{,}000 \\
X_3 &\le 1{,}500{,}000 \\
X_4 &\le 1{,}800{,}000 \\
X_3 + X_4 &\ge 0.55(X_1 + X_2 + X_3 + X_4) \\
X_1 &\ge 0.15(X_1 + X_2 + X_3 + X_4) \\
X_1 + X_2 + X_3 + X_4 &\le 5{,}000{,}000 \\
X_1, X_2, X_3, X_4 &\ge 0
\end{aligned}
$$

Portfolio Applications (con't)

- The optimal solution to the ICT is to make the following investments:

  X_1 = \$750,000

  X_2 = \$950,000

  X_3 = \$1,500,000

  X_4 = \$1,800,000

- The total interest earned with this plan is \$712,000.

Portfolio Applications (con't)

	A	B	C	D	E	F	G	H
1	**ICT Portfolio Selection**							
2								
3	**Variable**	**X1**	**X2**	**X3**	**X4**			
4	**Solution**	750000	950000	1500000	1800000	**Total Return**		
5	**Max. Return**	0.07	0.11	0.19	0.15	712000		
6								
7						**LHS**		**RHS**
8	**Trade**	1				750000	≤	1,000,000
9	**Bonds**		1			950000	≤	2,500,000
10	**Gold**			1		1500000	≤	1,500,000
11	**Construction**				1	1800000	≤	1,800,000
12	**Min. Gold+Const**	-0.55	-0.55	0.45	0.45	550000	≥	0
13	**Min. Trade**	0.85	-0.15	-0.15	-0.15	0	≥	0
14	**Total Invested**	1	1	1	1	5000000	≤	5000000

References

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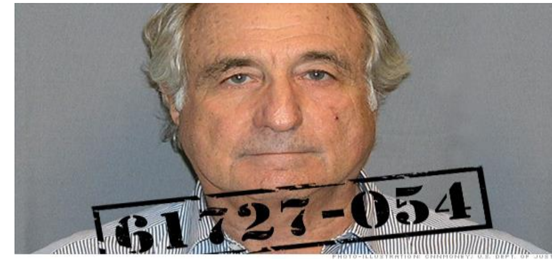
Homework

- Textbook Chapter 7, 8
- Quiz on these slides and Chapter 7 next session
- Discussion Questions to be answered: 1, 3, 5, 8, 9, 13 from Chapter 7
- Project Six →

Project 6



- Bernie Maddog could have had a balanced portfolio for his clients (instead of getting "caught up" in the ponzi scheme)
- Based on the investment choices shown in the following slide, find the optimal mix of investments, subject to these balancing rules:
  - Average risk is below 1.9
  - No more than 50% of portfolio in any one investment
  - At least 20% bonds (corp plus muni)
  - Growth at least a factor of 12

Investment Choices

Investment	Return	Risk	Growth
Muni Bonds	6	1.3	0
Corp Bonds	8	1.5	0
Common Growth Stock	5	1.9	15
Mutual Funds	7	1.7	8
Real Estate	15	2.7	20